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(U) FLIGHT MECHANICS OF A SATELLITE ROCKET

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DOUGLAS AIRCRAFT COMPANY, INC.

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PROJECT RAND

(AAF PROJECT MX-791)

FLIGHT MECHANICS OF A SATELLITE ROCKET

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RA-15021

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DOUGLAS AIRCRAFT COMPANY, INC.

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LIST OF FIGURES

*This initial external distribution list includes the distribution of all related technical reports on the satellite vehicle.

SUMMARY

As the result of extensive trajectory calculations it is found that when the hydrazine-oxygen propellant system is employed, a satellite carrying a payload of 500 pounds may be established on a periodic orbit by using a rocket of about 85,000 pounds initial gross weight. In order to achieve satisfactory duration an orbital height of 350 miles is recommended. The investigation shows that in this case a three stage rocket is best and that a maximum load factor of about 5 should be used for each stage. The study is presented in two parts. Part I consists of a presentation and simplified discussion of the various parameters upon which the trajectory performance depends. Arguments are given to show how, for all practical purposes, it is best to use independent staging, constant mass flow burning, and to use the same ν (propellant weight to gross weight ratio) and the same n (maximum load factor) for each rocket stage. The flight path control is accomplished by tilting the rocket which, since the axis of the motors corresponds essentially with the longitudinal axis of the rocket, provides a component of rocket motor thrust to give lift. Qualitative arguments are presented to show how it is best to have a trajectory which starts in the vertical, has a small amount of tilt beginning early in the flight, and has a long period of coasting occurring late in the trajectory. Finally a very useful simplified formula is derived giving the basic relation existing between the various trajectory parameters.

In Part II the accurate equations of motion are presented, and the method of integration is discussed. The ultimate aim of the calculations is to determine the optimum trajectory for a three stage rocket as based on the best tilt program, best coasting arrangement, and the best value for the maximum load factor. Some results are presented for different propellant systems, rockets having other than three stages, and for various orbital heights. Some attention is given to the questions of trajectory accuracy, orbital stability, and the descending trajectory.

LIST OF SYMBOLS

A = area
a = acceleration
a = semi-major axis of elliptical orbit
b = semi-minor axis of elliptical orbit
C_D = drag coefficient
C_L = lift coefficient
D = drag
E = energy
F = thrust
f = force
g = absolute acceleration of gravity at any height h
g_R = absolute acceleration of gravity at sea level = 32.199 ft/sec^2 at the equator
g_s = gravitational conversion factor = 32.174 ft/sec^2
h = vertical height above sea level
I = specific impulse
L = lift = force normal to trajectory
L_a = aerodynamic lift
m = mass
N = total number of rocket stages
N_r = total number of complete revolutions of the satellite body in its orbit to give a specified decrease in height
n = maximum load factor
Q = applied structural load
q = $1/2 \ \rho v^2$ = dynamic pressure
R = radius of the earth = 3963.34 miles at the equator
r = distance from the center of the earth to the height h; $r = R + h$
S_h = range measured with respect to a non-rotating coordinate system
S_h' = range measured with respect to a coordinate system which rotates with the earth
s = distance
T = total time
T_r = total time required for the number of revolutions N_r
t = time
t_b = length of burning period
U = energy constant
v = velocity referred to non-rotating coordinate system
v' = velocity referred to a coordinate system rotating with the earth
W = mg_s = standard value for weight based on the standard sea level value for gravity g_s = 32.174 ft/sec^2
W_i = total initial weight (gross weight) of any stage = $W_B + W_P + W_p + W_{L^*}$
W_j = total initial weight (gross weight) of a particular stage
W_B = basic weight = $W - W_{P^*} - W_{L^*}$. The basic weight includes structure, rocket motors, and all component parts except payload, rocket motor propellants, and auxilary fuel
W_{L^*} = payload of any stage
W_P = weight of rocket motor propellants at the beginning of a stage
W_p = weight of auxiliary fuel
W_{P^*} = weight of rocket motor propellants plus auxiliary fuel = $W_P + W_p$

$W_{L'}$ = payload of final stage = $W_L + W_I + W_Q$
W_L = payload of the satellite body
W_I = weight of the "intelligence"
W_Q = weight of the orbital power plant and its fuel
α = angle of tilt of rocket = angle between trajectory and longitudinal axis of rocket = angle of attack
α^* = effective angle of tilt
Δ = finite difference
ϵ = W_{p*}/W_p
θ = angle of inclination of the trajectory with the instantaneous horizontal when referred to a non-rotating coordinate system
θ' = angle of inclination of the trajectory with the instantaneous horizontal when referred to a coordinate system rotating with the earth
λ = non-dimensional time factor which specifies the beginning of tilt
ν = ratio of propellant weight to gross weight = W_p/W_i or $(W_p/W)_j$
ν^* = ratio of weight of propellants plus auxiliary fuel to gross weight = W_{p*}/W_i
ρ = mass density
τ = non-dimensional time factor which specifies the beginning of coasting
ϕ = polar angle at the center of the earth
Ω = angular velocity of rotation of the earth = 7.2911×10^{-5} radians/sec
ω = angular velocity of rotation when referred to a non-rotating coordinate system
ω' = angular velocity of rotation when referred to a coordinate system rotating with the earth

Subscripts

i refers to initial conditions
f refers to final conditions
t refers to instantaneous values of quantities variable with time when this is not otherwise understood.
F refers to an overall final condition
T refers to an overall total time
0 refers to a specified condition
orb refers to an orbit
req refers to a required condition
j refers to a particular stage

ERRATA

Page	Line	Reads	Should Read
2	18	...is given by the expression[(1), (5)]...	...is given by the expression[(2), (6)]...
22	11	...more exact performance equation, (38)...	...more exact performance equation, (41)...
26	37	...(mentioned in section 6)...	...(mentioned in section 8)...
29	7	...proceeding backward from E, will be indicated in C...	...proceeding backward from E, will be as indicated in C...
33	1	Eq.(54) should be mentioned...	Eq.(55) should be mentioned...
35	9	...This enables one to ascertain the the change in ν shape of the rocket...	...This enables one to ascertain the change in ν caused by a change in shape of the rocket...
55	Fig. 9 *Legend*	Weights determined according to methods outlined in ref.2	Weights determined according to methods outlined in ref.3
56	Fig. 10 *Legend*	...outlined in ref.2.	...outlined in ref.3.
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57	Fig. 12 *Legend*	...see eq.(19) ref.5.	...see eq.(19) ref.6.
57	Fig. 12A *Legend*	...see eq.(19) ref.5.	...see eq.(19) ref.6.
63	Fig. 17A *Legend*	See ref.2 for details...	See ref.3 for details...
65	Fig. 18 *Legend*	...based upon the methods of ref.2.	...based upon the methods of ref.3.
66	Fig. 19	...methods outlined in ref.2.	...methods outlined in ref.3.

FLIGHT MECHANICS OF A SATELLITE ROCKET

INTRODUCTION

The basic problem to be discussed in this report is that of establishing circular orbital motion about the earth in the equatorial plane with a rocket so that the rocket becomes a satellite of the earth, and which will therefore be referred to as a satellite rocket. Some of the general features of this problem have already received some attention in a preliminary report(1). However, since the present analysis has been organized in such a completely different manner, the presentation contained herein has been made complete in itself.

The establishment of a circular orbit about the earth requires that the satellite rocket be designed to attain a final orbital speed of the order of 25,000 ft/sec in order that the force of gravity acting on the rocket will be exactly balanced by the centrifugal force resulting from the orbital motion. Besides the problem of achieving the orbital speed there is the even more important problem of determining the best trajectory to use in order to establish the orbit with a rocket having the least possible gross weight, and it is this aspect of the problem which forms the basic investigation presented herein. The investigation is divided into two main parts. The first part deals with the subject in a somewhat general and approximate manner which, nevertheless, yields some important simplified results. The second part contains a detailed trajectory analysis together with the results of the complete trajectory study.

I. GENERAL CONSIDERATIONS OF PERFORMANCE AND DESIGN PARAMETERS OF A SATELLITE ROCKET

Before taking up the analysis of the flight mechanics of establishing circular orbital[1] motion with a satellite rocket, it will be advantageous to first discuss the problem from a more general standpoint in order to become familiar with the funda-

(1) For references see page 89.

[1] For footnote, see page 2.

mental parameters which govern the motion of the rocket along a trajectory. The basic problem is that of establishing the orbit with a rocket having the least possible gross weight, where the gross weight is defined as the total initial weight of the rocket, consisting of basic weight, propellant weight, payload, etc. The fundamentals of this problem can be discussed from some relatively simple concepts.

It is found that certain fundamental basic parameters connected with the structure and operation of the rocket may be chosen, and that these parameters have certain optimum values corresponding to a trajectory[1] of optimum shape such that the required gross weight is a minimum. The concepts and basic principles involved are discussed first in part I in a simplified and rather qualitative fashion in order that the more rigorous quantitative procedure in part II may be better understood. It is first necessary to introduce the concept of specific impulse.

1. The Specific Impulse

In rocket motor nomenclature, the materials which are burned are called the propellants and consist of fuel and oxidizer. The high speed exhaust gases resulting from combustion of the propellants produce the thrust which drives the rocket. It is readily shown by use of Euler's momentum theorem that the thrust F produced by a rocket motor is given by the expression [1], [6]

$$F = \int V_{ex} \frac{dm'_p}{dt} + \int (p_e - p_o)\, d\sigma \ , \tag{1}$$

where the integration extends over the exhaust area, and where

V_{ex} = axial component of exhaust velocity relative to the nozzle exit

$d\sigma$ = element of exhaust area

p_e = exhaust pressure

p_o = free-air pressure

$\frac{dm'_p}{dt}$ = element of mass of propellants flowing through the exhaust area of the nozzle per second (i.e., rate of mass flow of propellants through an element of exhaust area).

It is seen, in general, that unless there is complete expansion of the exhaust gases to give $p_e = p_o$, the thrust will consist of two parts, one part called the velocity

[1] The ascending trajectory, usually referred to simply as the trajectory, is defined as the *controlled* part of the flight path over which the rocket moves and in addition any section of uncontrolled flight followed by more controlled flight. In general this will consist of the flight path when either thrust or lift forces are operating plus the flight path during coasting. The orbit is defined as the path of the motion which is established at the end of the trajectory, a path which is nearly repeated after each revolution about the earth. In the case of descending motion, the corresponding flight path may be referred to as a descending trajectory even though there may be no control exercised on the motion.

thrust and a second part called the pressure thrust. In the case of the ideal rocket motor in which there is complete expansion of the exhaust flow and the exhaust velocity V_e has no radial component (i.e., one-dimensional), the thrust is simply

$$F = V_e \frac{dm_{p\,t}}{dt} \quad , \tag{2}$$

where $dm_{p\,t}/dt$ is the total rate of mass flow of propellants[2]. This equation shows how fundamentally the thrust depends upon the exhaust velocity.

The thrust may be increased through the term $dm_{p\,t}/dt$, since this may be made arbitrarily large simply by providing adequate means for delivering and burning the necessary amounts of propellants. This is not the case with the exhaust velocity which is more strictly a characteristic of the propellant used and is thus a parameter upon which the relative performance of different propellants might be based. For this purpose however it is customary to use a closely related quantity I known as the specific impulse which is defined by the relation

$$I = \frac{F}{g_s \dfrac{dm_{p\,t}}{dt}} \quad , \tag{3}$$

where g_s is the standard sea-level acceleration of gravity[3], and where I is expressed in units of pounds of thrust obtained per pound of propellants used per second. It will be found, Refs. (6) and (9) that the specific impulse may be computed in terms of the temperature and pressure in the combustion chamber, the ratio of the specific heats, the molecular weight of the ejected gases, and the amount of expansion in the exhaust nozzle. Thus when the specific impulse is known, the thrust is given very simply by

$$F = g_s \frac{dm_{p\,t}}{dt} I \quad . \tag{4}$$

2. Rocket Performance

Let us now consider in an elementary fashion the motion of a rocket along a trajectory defined in a stationary two-dimensional rectangular coordinate system as shown in Fig. 1. For simplicity it will be assumed at first that there are no aerodynamic forces or forces due to the earth's motion so that the only forces acting are gravity, the thrust of the rocket motor, and centrifugal forces. If v denotes the speed in the direction of the tangent to the trajectory, the equation of motion of the center of mass along the path (trajectory) is

$$m \frac{dv}{dt} = F - mg \sin \theta \ , \tag{5}$$

[2] It is convenient for the later developments to introduce the subscript t here to indicate that the quantity is variable with time.

[3] The value g_s is the standard value of gravity, 32.174 ft/sec^2, used in converting from slugs mass to pounds weight.

where m is the mass of the rocket and θ is the angle between the horizontal and the tangent to the trajectory (the angle of inclination of the trajectory). In the absence of lift forces, the corresponding equation of motion in the direction normal to the trajectory is

$$mv\frac{d\theta}{dt} = -mg\cos\theta + m\frac{v^2}{r}\cos\theta\ , \tag{6}$$

where r is the distance from the center of the earth to the rocket (see part II). In the equations of motion, g is the absolute value of the acceleration of gravity corresponding to the distance r of the vehicle from the center of the earth. Absolute gravity is distinguished from apparent gravity which contains the centrifugal force effect of the earth's rotation and which is less than the absolute value by this amount. Since the mass rate of change of the rocket is connected with the mass rate of propellant consumption by the relation[4]

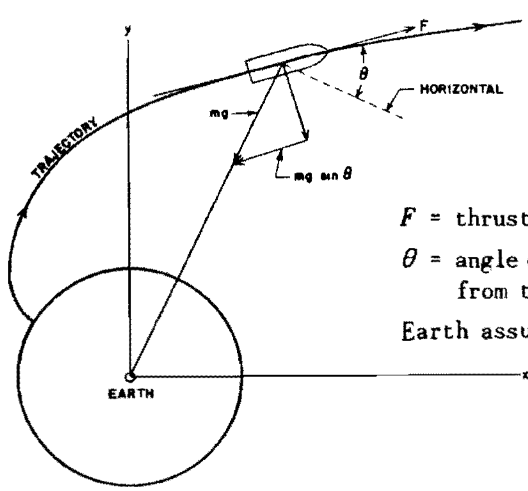
$$\frac{dm}{dt} = -\frac{dm_{pt}}{dt}\ ,$$

the specific impulse may be introduced by means of

$$F = -g_s\frac{dm}{dt}I\ , \tag{7}$$

where dm/dt is negative during burning, and Eq. (5) becomes

$$dv = -gI\frac{dm}{m} - g\sin\theta dt\ . \tag{8}$$



F = thrust of rocket motor.

θ = angle of inclination of trajectory from the horizontal.

Earth assumed not to rotate.

SCHEMATIC DIAGRAM TO ILLUSTRATE MOTION OF ROCKET ALONG A TRAJECTORY

FIG. I

[4] It is understood here, without using the subscript t, that m is a function of time.

Assuming as a first approximation that the specific impulse remains constant during the burning, the integration of this equation yields

$$\Delta v \equiv v_f - v_i = g_s I \log \frac{m_i}{m_f} - \int_{t_i}^{t_f} g \sin\theta \, dt \, , \tag{9}$$

where the subscript i indicates the initial conditions at the beginning of the burning period and the subscript f the final conditions at the end of the burning period. The length of the burning period is the difference $t_f - t_i$ which will be represented by t_b. This approximate equation emphasizes the two most fundamental parameters in rocket performance: the specific impulse, which depends in large part on the exhaust velocity, and the ratio of the total mass of the rocket at the beginning and end of the acceleration (burning), which depends upon the proportion of the total mass comprising propellant mass. It is evident that any increase that can be secured in the values of I and m_i/m_f will result in increased velocity gain of the rocket. Letting m_p denote the mass of propellants consumed during the burning period, the equation may be written

$$\Delta v = g_s I \log \frac{m_i}{m_i - m_p} - \int_{t_i}^{t_f} g \sin\theta \, dt \, . \tag{10}$$

Since the first term on the right is independent of time, it follows that the gain in velocity due to the thrust of the rocket motor for a given mass consumption of propellants is independent of the manner in which the burning takes place. The burning time t_b, on the other hand, is strictly dependent upon the manner of burning and here one may distinguish between two fundamental cases: (1) burning with $dm_{pt}/dt = -dm/dt =$ const. which by Eq. (4) gives operation at constant thrust assuming I does not vary, and (2) burning with $F/m =$ const. (i.e. $1/m \times dm_{pt}/dt = -1/m \times dm/dt =$ const.) which, by Eq. (5), gives operation at approximately constant acceleration if the secondary effects due to the inclination of the path are neglected. Before evaluating the burning time for these two cases of rocket-motor operation, a quantity called the load factor is introduced.

3. The Load Factor

A parameter n_t called the instantaneous load factor is now introduced by the defining equation

$$n_t g_s = \frac{dv}{dt} + g \sin\theta \; . \tag{11}$$

This definition follows automatically from Eq. (5),

$$\frac{F}{m} = \frac{dv}{dt} + g \sin\theta \, .$$

where it is convenient to express the total acceleration, or force per unit mass, $(dv/dt + g \sin \theta)$, in terms of the standard acceleration of gravity g_s. When the only forces present are those entering into Eq. (5), the definition then also satisfies the relation

$$n_t g_s = \frac{F}{m} . \tag{12}$$

The *maximum* load factor occurring in any burning period will be denoted by n without subscript where

$$n g_s = \left(\frac{dv}{dt} + g \sin \theta\right)_{max.} , \tag{13}$$

which, with the same restrictions as for Eq. (12), also satisfies the relation

$$n g_s = \left(\frac{F}{m}\right)_{max.} \tag{14}$$

The quantity ng_s is the maximum force, per unit mass of rocket, which is exerted on the structure in an axial direction opposite to the thrust force F. It is the sum of the axial inertia force and the axial component of the gravity force. The load exerted on the structure at any instant is given by the product $mn_t g_s$ and the maximum load will be $g_s\ (mn_t)_{max.}$ For operation with constant thrust it follows that $mn_t g_s$ = const. = F, and since in this case the load exerted on the structure will be constant during any particular burning period, there will be no maximum load, although there will be a maximum value n for n_t. On the other hand, for operation with F/m = const. there will exist, in general, in any particular burning period, a greatest value for the product $mn_t g_s$.

4. The Propellant-Gross Weight Parameter

In view of the fundamental importance of the ratio m_i/m_f occurring in Eq. (9), it is found convenient to introduce a propellant-gross weight parameter ν defined by

$$\nu = \frac{W_p}{W_i} , \tag{15}$$

where W_p is the weight of the propellants and W_i is the gross weight[5] of the rocket (see section 6). From the direct proportionality between weight and mass, it follows from the notation used in Eq. (10) that $\nu = W_p/W_i = m_p/m_i$ and Eq. (10) may be written

$$\Delta v = g_s I \log \frac{1}{1 - \nu} - \overline{g \sin \theta} \times t_b , \tag{16}$$

where t_b is the burning time and where for simplicity the integrand has been replaced by its mean value $\overline{g \sin \theta}$.

[5] For a single stage rocket the gross weight is defined as the total initial weight of the rocket including the weight of the propellants. From the definitions it is seen that W_p, m_p, W_i, and m_i are constants independent of time.

5. The Burning Times

We may now evaluate t_b in terms of the parameters I, n, and ν, for the two fundamental cases of burning.

Case 1.

Burning with dm/dt = const. (constant rate of mass flow of propellants) which, if I is constant, corresponds, by Eq. (7), to operation at constant thrust. In this case since F is constant, the value of F/m is a maximum at the end of the burning period when m is a minimum, and therefore Eq. (14), $ng_s = F/m_f$. The burning time is evaluated by integration of Eq. (7) which gives

$$t_b = \frac{g_s I}{F}(m_i - m_f) = \frac{g_s m_f}{F} I\left(\frac{m_i}{m_f} - 1\right) = \frac{g_s m_f}{F} I\left(\frac{W_i}{W_i - W_p} - 1\right) \; ..$$

Introducing the parameters n and ν, this may be written

$$t_b = \frac{I\,\nu}{n(1 - \nu)} \quad . \tag{17}$$

Case 2.

Burning with F/m = const. [$1/m\ (dm/dt)$ = const.] which corresponds approximately to operation with constant acceleration. In this case since F/m is constant, any value of F/m occurring during the burning period may be used in Eq. (13), and we have $ng_s = (F/m)_{max.} = F/m = F_f/m_f$. The burning time is evaluated by first dividing Eq. (7) by m giving

$$\frac{F}{m} = -\,g_s I \frac{1}{m}\frac{dm}{dt} \quad .$$

Assuming constant I, integration of this results in the expression

$$t_b = \frac{g_s m}{F} I \log\left(\frac{m_i}{m_f}\right) = \frac{g_s m}{F} I \log\left(\frac{W_i}{W_i - W_p}\right) \cdot$$

Introducing the parameters n and ν, this may be written

$$t_b = \frac{I}{n}\log\frac{1}{1 - \nu} \quad . \tag{18}$$

It may be pointed out that for the same values of I, n, and ν, the value of t_b given by Eq. (18) will be less than that given by Eq. (17) for the range of values of the parameter ν of practical importance.

6. Orbital Speed and Staging

The problem of establishing a rocket on a circular orbit encompassing the earth is primarily one of accelerating the rocket to a high enough velocity, called the orbital velocity, which is determined by the condition that the centrifugal force is exactly balanced by the gravity force. The problem of guiding the rocket into the orbital path is secondary as far as the requirements of performance and design are concerned. Let us consider the limitations on acceleration imposed by structural considerations. Using the following notation,

W_B = basic weight, which includes all component parts except payload, rocket motor propellants, and auxiliary fuel.

W_P = total initial weight of rocket motor propellants

W_{L*} = weight of payload

W_p = total initial weight of auxiliary fuel

W_i = total initial weight, or gross weight = $W_B + W_P + W_p + W_{L*}$,

the propellant-gross weight parameter may then be written

$$\nu = 1 - \frac{W_B}{W_i} - \frac{W_{L*}}{W_i} - \frac{W_p}{W_i} \quad . \tag{19}$$

Since W_p/W_i is extremely small and relatively unimportant compared to the other terms in this equation, it will be neglected for the present and the parameter ν will be treated on the basis of the relation

$$\nu = 1 - \frac{W_B}{W_i} - \frac{W_{L*}}{W_i} \quad . \tag{20}$$

The auxiliary fuel will be brought into the discussion later, Eqs. (77) and (78), where a quantity ν^* is defined in accordance with Eq. (19). Although large acceleration gives high performance, the associated large value of the load factor necessitates a heavier structure with a resulting higher value of the ratio W_B/W_i, and, as shown in the Structure and Weight Report[3], in order to obtain optimum performance there is a limitation on how small W_B/W_i may be. Thus, since W_{L*} is constant, the limitation on W_B/W_i sets a limit to the largest value permissible for ν and therefore to the largest acceleration attainable. Likewise, as shown in the Liquid Propellant Report[9], there is a limitation as to how high a value can be attained for I.

Owing to these limitations on I and ν, it is seen from Eq. (16) that there is a corresponding limitation on how large a value may be obtained for Δv. Since the main problem is to establish an orbit with a rocket having the least possible gross weight, it follows that we must seek conditions (both launching and orbital) which will make the required Δv as small as possible. Accordingly, not only should the orbital velocity required be as small as possible, but also it would be highly desirable to launch the rocket with an initial velocity as large as possible. The idea of launching anything as large and heavy as the satellite rocket with an initial velocity is, of course, highly impractical and is dismissed immediately. Also, there is very little choice in, or control over, the magnitude of the required orbital velocity. However, we do have a choice in selecting the launching site, and in this respect it is found highly advantageous to have the launching take place at the equator with the trajectory directed eastward.

Although the rotation of the earth has been neglected in the simplified discussion of part I, it is evident that when the launching takes place at the equator and the trajectory is directed eastward, the fullest possible advantage is taken of the linear velocity at the surface of the earth due to earth's rotation. Launching the rocket in this manner at the equator makes the required change in velocity Δv as small as possible and the orbital velocity required with respect to the rotating earth as small as possible. Since the linear velocity at the equator is 1525 ft/sec, the required velocity Δv is less by this amount than it would be for a non-rotating earth. This saving in required velocity change results in an important saving in the required gross weight of the rocket. The importance of this saving may be readily appreciated by the fact that when the rocket is launched at the equator, the gross weight required is about 30 percent less than if it were launched at one of the poles. Launching at the equator also has an important additional advantage, for, since the orbit will then lie in the equatorial plane for which there are no north-south components of the Coriolis' force, it is then possible to maintain orbital motion in a fixed plane relative to the earth without the application of any forces to control the path.

In view of the important advantages accruing from an equatorial launching of the rocket with its trajectory directed toward the east, this type of launching will be assumed in all succeeding discussion. However, in spite of the saving in gross weight resulting from an equatorial launching, it is still found that the largest value practically attainable for Δv is not sufficient to accelerate a single rocket up to orbital speed.

The only practical way of overcoming this shortcoming is to proceed on the basis of a staged rocket (multi-stage rocket), that is, a rocket which contains two or more rocket motor units which may be operated either simultaneously or consecutively and which are discarded when their propellants are consumed.

7. Independent and Dependent Staging

There are two basic methods of staging which may be referred to as independent staging and dependent staging. With independent staging the rocket motors are operated consecutively. As an illustration consider a two stage rocket. The primary rocket, which is the one to be established on the circular orbit, is carried along over part of the trajectory as the payload of a larger secondary rocket which furnishes the acceleration over the first part of the trajectory. When the secondary rocket has exhausted its propellants, and hence its usefulness, it is discarded; the primary rocket then continues to accelerate further under its own power, adding its own velocity increase to that imparted by the secondary rocket. In general, independent staging consists of an assembly of progressively smaller rockets each of which is carried as the payload of the preceding rocket.

As an illustration of dependent staging, consider a three stage rocket. Initially this will contain three rocket motors which are all in operation over the first part of the trajectory. The arrangement is such that after a first part of the trajectory has been covered, one of the motors and its propellant tanks are discarded and the rocket continues with only two motors operating. Later a second motor and its propellant tanks are discarded, and the final stage continues with thrust supplied by only the single remaining motor. Thus the motor in the final stage operates continuously over the whole trajectory.

In order to make a choice as to which of the two methods of staging is preferable, consider two staged rockets, one with dependent staging and the other with independent staging. Assume that both rockets use the same type propellants and rocket motors (and therefore have the same specific impulse I), have the same structural load factors and structural weights $(W_B/W)_i$, and also that the thrust program (variation of thrust with time) is the same for each. Although not strictly justified, if it be assumed at first that the rocket weight per stage is the same for both cases, it then follows from this, plus the additional conditions assumed above, that the trajectory and flight conditions will be the same in both cases although the manner in which they are produced will be different, since the rocket arrangements are different. Actually the combined weight of the rocket motors may be made less when dependent staging is used than when independent staging is used, and on this basis alone it would appear better to use dependent staging since this would give a smaller initial gross weight. However, owing to the additional auxiliary equipment which would be required by this design, the actual net saving in weight, if any, would be small; and since a dependent staging design would be a much more complex one, it has been decided to adopt independent staging for the satellite rocket.

8. Performance With Multiple Independent Staging

Having adopted the independent staging scheme, the performance equation (16) may be considered again, this time from the staging point of view. In the case of multiple staging the parameter ν is defined for each stage by

$$\nu_j = \left(\frac{W_P}{W}\right)_j = \frac{\text{weight of propellants used in the stage}}{\text{gross weight of the stage}},$$

where the subscript j indicates the particular stage[6]. With this notation it is no longer necessary to use the subscript i to indicate the initial total (gross) weight. Consider for example a three stage rocket, where the stages are indicated by the subscripts $_1$, $_2$, and $_3$. With independent staging the payload of one stage is the gross weight of the following stage. Thus

$$\left(\frac{W_{L'}}{W}\right)_1 = \frac{W_2}{W_1}, \left(\frac{W_{L'}}{W}\right)_2 = \frac{W_3}{W_2}, \left(\frac{W_{L'}}{W}\right)_3 = \frac{W_{L'}}{W_3}, \tag{21}$$

where the symbol $W_{L'}$ without numerical subscript is used to denote the payload in the final stage. From Eq. (20) it then follows that

[6] When a particular stage is indicated by the subscript j, it will not be necessary to use any other subscript (such as i) to indicate that it is the gross weight of the stage which is involved. Thus in the expression $(W_P/W)_j$, it is to be understood that W is the gross weight of the jth stage. The expression W_i (or m_i) is used to indicate the gross weight in a more general sense not pertaining to any particular stage. When W (or m) has no associated subscript, it is then simply the total weight and is variable with time. When it is necessary to indicate a quantity, in a particular stage, which is also variable with time, the subscript t will be used together with j as indicated by W_{jt} (or m_{jt}), for example.

$$
\begin{aligned}
1 - \nu_1 &= \left(\frac{W_B}{W}\right)_1 + \left(\frac{W_L^*}{W}\right)_1 = \left(\frac{W_B}{W}\right)_1 + \frac{W_2}{W_1} , \\
1 - \nu_2 &= \left(\frac{W_B}{W}\right)_2 + \left(\frac{W_L^*}{W}\right)_2 = \left(\frac{W_B}{W}\right)_2 + \frac{W_3}{W_2} , \\
1 - \nu_3 &= \left(\frac{W_B}{W}\right)_3 + \left(\frac{W_L'}{W}\right)_3 = \left(\frac{W_B}{W}\right)_3 + \frac{W_L'}{W_3} .
\end{aligned}
\tag{22}
$$

This last expression may be written in the form

$$
1 - \nu_3 = \left(\frac{W_B}{W}\right)_3 + \frac{W_L'}{W_1} \times \frac{W_1}{W_2} \times \frac{W_2}{W_3} ,
$$

which, by use of relations (22), may finally be expressed in the form

$$
\frac{W_L'}{W_1} = \left[1 - \nu_1 - \left(\frac{W_B}{W}\right)_1\right]\left[1 - \nu_2 - \left(\frac{W_B}{W}\right)_2\right]\left[1 - \nu_3 - \left(\frac{W_B}{W}\right)_3\right] . \tag{23}
$$

When ν and W_B/W are the same for all stages this may be written

$$
\left(\frac{W_L'}{W_1}\right)^{\frac{1}{3}} = \left[1 - \nu - \left(\frac{W_B}{W}\right)_1\right] , \tag{24}
$$

where W_1 is the initial gross weight. In the general case for N stages this gives

$$
\left[1 - \nu - \left(\frac{W_B}{W}\right)_1\right] = \left(\frac{W_L'}{W_1}\right)^{\frac{1}{N}} . \tag{25}
$$

For a staged rocket, the performance equation (16) may now be written

$$
\Delta v \equiv \sum_{j=1}^{N} \Delta v_j = g_s I \sum_{j=1}^{N} \log \frac{1}{1 - \nu_j} - \sum_{j=1}^{N} \overline{g_j \sin \theta_j} \times t_{b_j} , \tag{26}
$$

where the summation extends over all of the stages.

If the rocket starts from rest and attains its final velocity v_F without coasting (see section 13), it follows that $\Delta v = v_F$. If coasting occurs over part of the trajectory and the change in velocity during coasting is Δv_c, then $\Delta v = v_F - \Delta v_c$, where Δv_c will be small compared to Δv and is negative. If the final velocity attained corresponds to an orbital velocity $v_{orb.}$, which will always be the desired condition, then $v_F = v_{orb}$. All discussions of Δv are to be thought of on this basis.

Consider a fixed value given for Δv of such magnitude, including any coasting effect that is present, that it will permit the attainment of an orbital velocity, i.e., $\Delta v = v_{orb} - \Delta v_c$. It will now be shown that when $\overline{(W_B/W)_j}$ and $\overline{g_j \sin\theta_j}$ are the same in all stages, i.e., $(W_B/W)_j = W_B/W_i$, $\overline{g_j \sin\theta_j} = \overline{g \sin\theta}$, a minimum value results for the initial gross weight W_i provided ν_j has the same value for each stage. The method by which this is proved will be illustrated by treating the case of a three stage rocket. Assuming $(W_B/W)_j = W_B/W_i = (W_B/W)_1 =$ const. in Eq. (23), form the derivative $d(W_{L'}/W_1)\, d\nu_1$ holding ν_2 constant, obtaining

$$\frac{1}{\left[1 - \nu_2 - \left(\frac{W_B}{W}\right)_1\right]} \frac{d\left(\frac{W_{L'}}{W_1}\right)}{d\nu_1} = -\left[1 - \nu_3 - \left(\frac{W_B}{W}\right)_1\right] - \left[1 - \nu_1 - \left(\frac{W_B}{W}\right)_1\right] \frac{d\nu_3}{d\nu_1} .$$

Setting this equal to zero to satisfy the condition for a maximum in $W_{L'}/W_1$ (a minimum in W_1), we have

$$\frac{d\nu_3}{d\nu_1} = -\frac{1 - \nu_3 - \left(\frac{W_B}{W}\right)_1}{1 - \nu_1 - \left(\frac{W_B}{W}\right)_1} . \tag{27}$$

For a three stage rocket with $\overline{g \sin\theta} =$ const., Eq. (26) becomes

$$\Delta v = g_s I \left[\log\frac{1}{1-\nu_1} + \log\frac{1}{1-\nu_2} + \log\frac{1}{1-\nu_3}\right]$$
$$- \overline{g \sin\theta} \left(t_{b_1} + t_{b_2} + t_{b_3}\right) .$$

Performing the differentiation $d/d\nu_1$, using $d/d\nu_1 = d/d\nu_3 \times d\nu_3/d\nu_1$ and remembering that Δv is considered as having a given fixed value results in the relation

$$\frac{d\nu_3}{d\nu_1} = -\frac{\left(\frac{g_s I}{1-\nu_1} - \overline{g \sin\theta}\,\frac{dt_{b_1}}{d\nu_1}\right)}{\left(\frac{g_s I}{1-\nu_3} - \overline{g \sin\theta}\,\frac{dt_{b_3}}{d\nu_3}\right)} . \tag{28}$$

It is evident that the equivalent Eqs. (27) and (28) are compatible only if $\nu_3 = \nu_1$, and therefore this is the condition which must be satisfied if the gross weight

W_1 is to be a minimum. In a similar manner it can be shown that minimum gross weight requires that $\nu_2 = \nu_3$. Hence, if there are N stages,

$$\nu_1 = \nu_2 = \nu_3 = \cdots = \nu_N = \nu \tag{29}$$

is the required condition for minimum gross weight W_1, in the special case that $g \sin\theta$ and W_B/W are the same in all stages. This is an important result since it indicates that even in the general case, where $\overline{g_j \sin\theta_j}$ and $(W_B/W)_j$ vary with staging, the values of ν_j are not expected to be markedly different if the condition for minimum gross weight W_1 is to be fulfilled. What is finally desired, of course, is the optimum distribution of ν_j with staging to give a minimum gross weight W_1 in the general case where the quantities in Eq. (26) vary from stage to stage.

Actually $\overline{\sin\theta_j}$ will vary considerably from stage to stage. It is apparent that in the first stage where the trajectory is more nearly vertical, the term $\overline{g_j \sin\theta_j} \times t_{b_j}$ will be relatively larger, compared to the thrust term $g_s\, I \log 1/1 - \nu_j$, than in the second and remaining stages. In this connection suppose that there is a given weight of propellants available for burning. Then it is seen from Eq. (26) that the greatest increase in velocity Δv will result when relatively greater amounts of the propellants are burned over those portions of the trajectory where $\overline{g_j \sin\theta_j}$ is relatively small. Therefore, as can be readily shown analytically, the optimum distribution of ν_j will always be such that $\nu_1 < \nu_2 < \nu_3 \ldots < \nu_N$. In other words, it is most efficient to burn relatively greater amounts of the propellants (in terms of the ratio ν) in those stages where the burning can produce the greatest velocity increase as far as the retarding effect of gravity is concerned, that is, where the trajectory is relatively horizontal. In some preliminary investigations of an approximate nature it appeared that a decrease in gross weight amounting to about 10 per cent could be achieved by a proper staging of ν. However when a more accurate study was made, which included the effects of changes in shape of trajectory and variations in the values of $(W_B/W)_j$, the actual decrease in gross weight which could be achieved by means of a staging of ν turned out to be less than 5 per cent. The investigation of the optimum distribution of ν_j among the various stages (optimum staging of ν) has therefore proceeded far enough to show that no appreciable decrease in gross weight can be achieved by this means. Thus since the condition $\nu_j = \nu =$ constant is very nearly optimum, the flight mechanics investigation contained in this report has been carried out on this basis, i.e., $\nu =$ constant as far as staging is concerned.

Aside from the fact that $\nu_j =$ constant represents practically optimum conditions, a further important concept expressed by Eq. (23) should be emphasized. Suppose $\nu_1 = \nu_2 = \nu_3$, then since W_L, has a fixed value, it follows from Eq. (23) that when the $(W_B/W)_j$ are fixed, the initial gross weight W_1 will be a minimum when ν is a minimum. It has been found from structural calculations[3] that, as far as the variation of the important parameters (other than the load factor n) which determine the trajectory is concerned, the values $(W_B/W)_j$ remain practically fixed and are approximately equal to each other, i.e., as indicated approximately by Eq. (24). Therefore, the determination of these parameters to give minimum ν is a satisfactory approach to the determination of minimum W_1. On the other hand, the variation of the maximum load factor n has just as important an effect in determining the structure[3], and therefore $(W_B/W)_j$, as it does in determining the trajectory. The determination of the minimum required value for W_1, which in most cases corresponds to the problem of finding the minimum value for ν, constitutes the basic investigation of this report. The details of the calculations required to solve this problem are discussed in Part II.

Assuming the values ν_j to be independent of staging and equal to ν, the performance equation (26) may be written

$$\Delta v \equiv \sum_{j=1}^{N} \Delta v_j = N g_s I \log \frac{1}{1-\nu} - \sum_{j=1}^{N} \overline{g_j \sin \theta_j} \times t_{b_j} . \qquad (30)$$

On the basis of staging, the expressions (17) and (18) for the burning times are written

$$t_{b_j} = \frac{I \nu}{n_j (1-\nu)} , \qquad (31)$$

for burning with dm/dt = constant, and

$$t_{b_j} = \frac{I}{n_j} \log \frac{1}{1-\nu} \qquad (32)$$

for burning with F/m = constant.

9. Qualitative Analysis of the Relative Optimum Staging Values of n_j and the Best Method of Burning to Employ for Obtaining These Values

The overall problem to be discussed in this section is that of determining the best method to employ for burning the propellants during flight. The manner (including rate) in which the burning takes place obviously determines the burning time t_{b_j}, which by Eq. (30), for constant I and ν, determines the trajectory performance. Furthermore, by Eqs. (31) and (32), it is seen that t_{b_j} also determines n_j and therefore essentially determines the structural loads which the rocket must be designed to withstand. This, of course, determines the weight of the rocket structure. Thus, it is seen that the determination of the best method of burning, through the burning time t_{b_j}, is the fundamental basic problem involved since it is fundamental in determining trajectory performance, and also determines the structural strength (weight) which the rocket must have. In fact, since weight and trajectory performance are closely related, the best method of burning is that which gives the least initial gross weight W_1 with regard to both structural weight and trajectory performance. The argument presented here emphasizes the flight mechanical point of view, while a similar argument given in Ref. 3 emphasizes the structural point of view.

Although there are several different ways in which the burning problem could be analyzed and discussed, it is convenient, and at the same time results in no loss in generality, to separate the problem into two parts. Using this type of approach we first seek the relative magnitude of the best (optimum) values for n_j, where the best n_j are determined from both structural and flight mechanical considerations, and where, in this first part of the analysis, the manner in which the burning must take place in order to give the best n_j is specified only to the extent that the same type of burning is assumed to take place in each stage.

Having this result, the second part of the analysis proceeds to compare different methods of burning, on the basis that the burning always leads to the same (best) values for the n_j, in order to determine which type of burning is best to give the least gross weight for the rocket. In this way the burning is always discussed on the basis that it gives the same value for the applied load on the structure (and also the same n_j) at the end of a particular burning period, although the loads at the beginning of the burning period may be equal to or greater than the load at the end of the period. The relative applied loads corresponding to the relative values found for n_j are discussed on the basis of two particular types of burning, burning with dm/dt = const. and burning with F/m = const. These two types of burning are sufficiently diverse to include within them any other type of burning which would be of practical interest.

Considering the trajectory performance Eq. (30) together with expressions (31) and (32) for the burning times, it will be observed that the larger the value of the maximum load factors n_j, the smaller will be the value obtained for ν when Δv is held constant. In fact, from this point of view, when $g_j \sin \theta_j$ is different from zero the least value for ν would result when $n_j = \infty$, that is, when $t_{b_j} = 0$. On the other hand when n_j increases, strength considerations require that $(W_B/W)_j$ increase, since the structure must be made heavier. Accordingly, for a *given* value of Δv, it is seen that the relation between the parameters ν, $(W_B/W)_j$, and n_j is such that an optimum value must exist for n_j, an optimum value always being understood as a value which leads to a minimum value for the gross weight W_1. Since it is important that these various optimum conditions be clearly understood, the following illustration may be helpful.

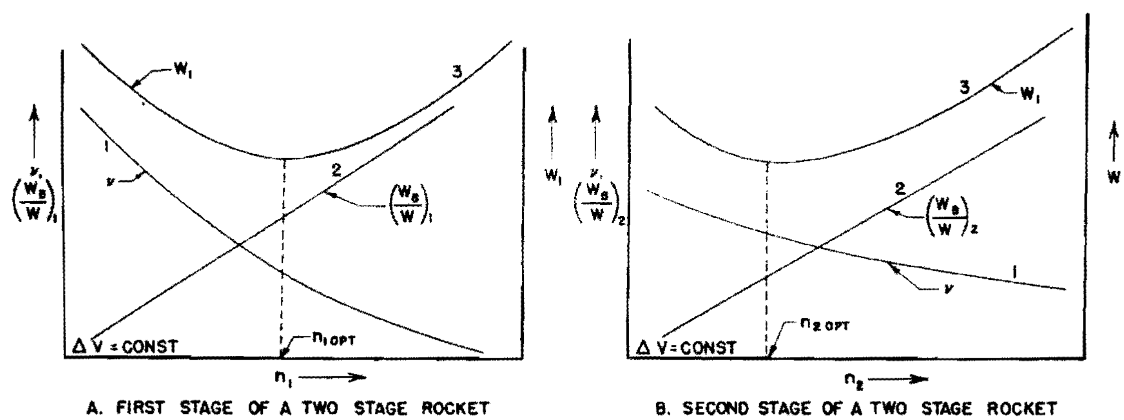
First of all it should again be pointed out that, since the rocket is to be established on a circular orbit at a given height above the surface of the earth, the necessary total change in speed $\Delta v \equiv \sum_{j=1}^{N} \Delta v_j$ to accomplish this is approximately the orbital speed itself which, together with Δv, will be considered as constant in the following treatment. Considering for the moment a two stage rocket with $(W_B/W)_j$ the same for each stage, we have from Eq. (25) that

$$\frac{W_{L'}}{W_1} = \left[1 - \nu - \left(\frac{W_B}{W}\right)_1\right]^2 = \left\{1 - \left[\nu + \left(\frac{W_B}{W}\right)_1\right]\right\}^2 .$$

Since $W_{L'}$ has a constant fixed value, it is clear that W_1 will be a minimum (in this particular case with $\nu_j + (W_B/W)_j$ = constant with staging) when the sum $\nu + (W_B/W)_1$ is a minimum. Now when n increases, while Δv remains constant, the orbital velocity is attained in a shorter burning time and therefore, from this effect alone, a smaller value of ν would be required. An increase in n, on the other hand, will require an increase in weight of structure per unit of gross weight, that is, an increase in the ratio $(W_B/W)_1$. As a result of these oppositely directed effects of change in n, it is apparent that a minimum value must exist for $\nu + (W_B/W)_1$ and therefore also for W_1, and the problem is to find the value of n which gives this minimum value. Consider now the more general case where $(W_B/W)_j$ may vary with staging. Referring again, for simplicity, to a two stage rocket we have

$$\frac{W_L{}'}{W_1} = \left\{1 - \left[\nu + \left(\frac{W_B}{W}\right)_1\right]\right\} \left\{1 - \left[\nu + \left(\frac{W_B}{W}\right)_2\right]\right\},$$

where it is now desired to find the values of n for each stage, independent of the other stages, which will result in a minimum value for the gross weight W_1 of the first stage. The manner in which this is accomplished is illustrated, for example, in (A) of Fig. 2 which shows schematically a typical variation of $(W_B/W)_1$, ν, and W_1 with n_1. The optimum values of n_j are defined as those values which lead to a minimum value for the gross weight W_1 of the first stage.



SCHEMATIC DIAGRAM TO ILLUSTRATE THE OPTIMUM STAGING OF n. (SEE REF. 3)

FIG. 2

The question may now be asked how the optimum value of n will vary with staging. This is illustrated for a two stage rocket in Fig. 2 where (A) represents the first stage and (B) the second stage. It is shown in the Structure and Weight Report[3] that in going to a higher stage, from the first to the second for example the variation of $(W_B/W)_1$ with n becomes smaller so that curve 2 of (B) has a slightly smaller slope for a given n than curve 2 of (A). It is also found, because of the much smaller $\overline{\sin\theta}$, that the variation of ν with n not only becomes smaller in going to a higher stage but that the decrease is much more pronounced than for the W_B/W variation. Thus curve 1 of (B) has a considerably smaller slope for a given n than curve 1 of (A). The net result of this is to give a smaller value for optimum n_2 than for optimum n_1, i.e., $n_{2_{opt}} < n_{1_{opt}}$. In general it is found that

$$n_{1_{opt}} > n_{2_{opt}} > n_{3_{opt}} > \cdot \cdot \cdot \cdot > n_{N_{opt}} \quad . \tag{33}$$

It must be emphasized that this result has been based on a hypothetical or ideal concept in which the fact that the rocket stages are carried within each other has been completely disregarded. When this necessary restriction is imposed, the relations (33) can no longer be completely fulfilled, as will become apparent from the discussion which follows.

For a three stage rocket, for example, the second and third stages are carried by the first stage, and the third stage is carried by the second stage. Thus any stage must undergo, at the very least, the maximum acceleration of any of the preceding stages and in particular that of the first stage.

Suppose that the optimum values of n have been determined for a three stage rocket as specified by the previous discussion (each stage considered independently of the others), and the staging of n is optimum on this basis so that

$$n_{3_{opt}} < n_{2_{opt}} < n_{1_{opt}} \ , \tag{34}$$

where the n_{opt} are the maximum load factors which are now assumed to exist in the different burning periods as far as trajectory performance, Eqs. (30) to (32), is concerned. In view of the fact that each stage is to be carried along over part of the trajectory within a preceding stage, consider the maximum structural loads to which each of the individual rockets will actually be subjected when motion over the entire trajectory is taken into account. Now the maximum applied load $Q_{3_{max}}$ to which the structure of the third stage rocket is subjected is given, in general, by $Q_{3_{max}} = g_s\,(m_{3t}n_{3t})_{max}$, where the subscript t refers to instantaneous values which are variable with time and which may occur over any part of the trajectory[6]. Similarly, the maximum load endured by the structure of the second stage is expressed by $Q_{2_{max}} = g_s(m_{2t}n_{2t})_{max}$. The load Q_{31} imposed on the structure of the third stage rocket during its travel with the first stage will be $Q_{31} = g_s m_{3i} n_{1_{opt}}$, where m_{3i} indicates the total initial mass of the third stage rocket with full propellant tanks.

Since, when $n_3 = n_{3_{opt}}$, the maximum load during the third burning period would never exceed $g_s m_{3i} n_{3_{opt}}$, it follows from relations (34) and $Q_{31} = g_s m_{3i} n_{1_{opt}}$ that, as far as the applied load which the structure of the third stage rocket must withstand is concerned, the structure of stage 3 must be made heavier than would be indicated simply by the optimum staging value $n_{3_{opt}}$ alone. Thus since the structure must be made at least as heavy as that required by the loading Q_{31}, the value for the maximum load factor n_3 in the third burning period may therefore be chosen greater than $n_{3_{opt}}$ without entailing any increase in structural weight. Similarly for the second stage structure the maximum load applied over the first stage of the trajectory is $Q_{21} =$

[6] See footnote on page 10.

$g_s m_{2i} n_{2opt}$. Since, for operation with $n_2 = n_{2opt}$, the maximum load during the second burning period would never exceed $g_s m_{2i} n_{2opt}$, but since the applied load Q_{21} is already greater than this by virtue of Eq. (34), it follows that the second stage structure must be made heavier than would be indicated by the value n_{2opt} alone. Therefore, as in the case of the third stage, the value for the maximum load factor n_2 in the second burning period may be chosen greater than n_{2opt} without entailing any increase in the structural strength (weight) which is required by the applied load Q_{21}. In view of these results that n_2 and n_3 may be chosen larger than n_{2opt}, n_{3opt} without entailing any increase in structural weight, the question immediately arises as to what are the best values to use for n_2 and n_3.

It has already been indicated in connection with Eq. (30) that, when the structural weight is fixed, the trajectory performance improves with an increase in the maximum load factor n. That is, if there is no increase in structural weight involved, it is always better from the trajectory standpoint to use the largest possible value for n, since, Eqs. (30) and (32), this will lead to a smaller retarding effect due to gravity. Accordingly, it is desirable to make n_2 and n_3 as much larger than n_{2opt}, n_{3opt} as is possible without necessitating any increase in structural weight.

To discuss n_2 and n_3 further requires a consideration of the load factors n_{2i}, n_{3i} at the beginning of the second and third burning periods, since, as can be shown from Eq. (12), for the two types of burning being considered here, the maximum applied load in any burning period always occurs at the beginning of the period. Thus, if there is to be no increase in structural weight beyond that required by the loads Q_{21} and Q_{31}, we must have $g_s m_{2i} n_{2i} \leqq g_s m_{2i} n_{1opt}$ and $g_s m_{3i} n_{3i} \leqq g_s m_{3i} n_{1opt}$, that is, $n_{2i} \leqq n_{1opt}$ and $n_{3i} \leqq n_{1opt}$. Furthermore, in order to satisfy the condition $g_s (m_{3t} n_{3t})_{max} \leqq g_s m_{3i} n_{1opt}$, it is evident that n_2 must be $\leqq n_{1opt}$. Thus, since from trajectory considerations the best value for n_2 is the largest value, the value $n_2 = n_{1opt}$ is adopted for the second stage burning period.

Since stage 3 is carried by stage 2, the additional condition must be satisfied that $m_{3i} n_{3i} \leqq m_{3i} n_2$. Hence the conditions which must be satisfied during the third stage burning period are

$$m_{3f} n_{3opt} \leqq m_{3f} n_3 \leqq m_{3i} n_{3i} \leqq m_{3i} n_2, \text{ or, using the adopted value}$$

$$n_2 = n_{1opt},$$

$$n_{3opt} \leqq n_3 \leqq \frac{m_{3i}}{m_{3f}} n_{3i} \leqq \frac{m_{3i}}{m_{3f}} n_2 = \frac{m_{3i}}{m_{3f}} n_{1opt} \,. \tag{35}$$

It is evident that n_3 may be chosen greater than n_{1opt} if it is desired, provided n_{3i} does not exceed n_{1opt}. However, the advantage accruing from such a choice is so small as far as trajectory performance is concerned, because of the small effect of gravity in this burning period, that $n_3 = n_{1opt}$ is adopted and therefore the performance of the satellite rocket will be based on the adopted values

$$n_3 = n_2 = n_{1opt} \,. \tag{36}$$

The determination of the value of n_{1opt} is discussed later.

The discussion above is illustrated by the diagrams in Fig. 3 for the two cases of burning, constant mass flow (dm/dt = const.) and constant F/m ($1/m\ dm/dt$ = const.). For the simplified conditions of Eq. (12) the instantaneous load factor n_t is given by $g_s n_t = F/m$. For burning with constant rate of mass flow of propellants, and with I assumed constant, the thrust F is constant and therefore the applied load $Q = g_s m n_t$ is constant and equal to F. Since F is constant, n_t varies from its lowest value at the end of the period. For burning with F/m = constant, the value of n_t remains constant during the burning period, and since the condition $1/m\ dm/dt$ = const. gives an exponential variation of m, the applied load $Q = g_s m n_t$ therefore varies exponentially from its greatest value at the beginning of the burning period to its least value at the end of the period.

It will be noted that the value of n_t at the end of a burning period is the same for both types of burning. This follows in accordance with the basic method of attack employed as discussed at the beginning of the section.

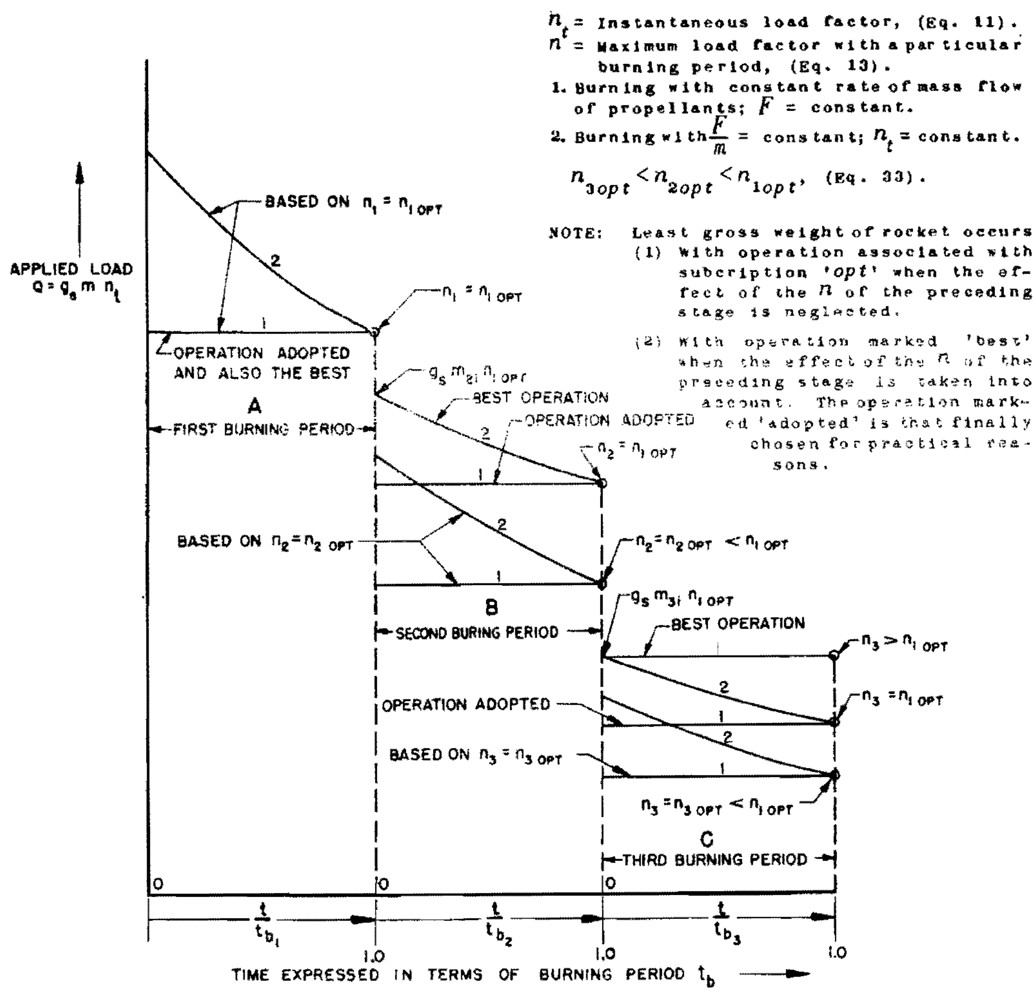


DIAGRAM TO ILLUSTRATE THE VARIATION WITH STAGING OF THE APPLIED STRUCTURAL LOAD FOR TWO BASIC METHODS OF BURNING

FIG. 3

The question of the best manner in which to have the burning occur may be answered at this point in order to simplify succeeding discussions. It was pointed out, Eq. (18), that burning with F/m remaining constant gives a shorter burning time than for burning with constant rate of propellant mass flow (dm/dt = constant); and considering the gravity term in the performance equation (30), it would appear better to have burning with F/m = constant. For a three stage rocket, for example, it is probably best to employ this type of burning in the second stage. For, in view of the adopted value $n_2 = n_{1_{opt}}$, the structural strength is adequate for both types of burning, but since burning with F/m = const. has the shorter burning time, this type of burning is to be preferred. On the other hand, the length of the third stage burning period has but little effect upon the trajectory, and the main consideration is one of structural weight, and burning with dm/dt = const. is recommended. In the first stage the applied load $Q_{1t} = g_s m_{1t} n_{1t}$ which the structure must withstand is, for the same value of n_1, less for burning with dm/dt = const. than for burning with F/m = const. Also, as shown in the structure investigation[3], the saving in gross weight W_1 by using dm/dt = const. is greater (for structural reasons) than the increase in gross weight resulting (for trajectory reasons) from the use of dm/dt = const. Therefore, in the first stage, burning with dm/dt = const. is recommended.

Calculations based on burning with F/m = constant in the second stage indicate a saving of 6 to 8 per cent in gross weight, while the corresponding saving in the third stage is considerably less. Actually it is questionable whether there would be any saving at all, since burning with F/m = constant would require additional equipment to produce variations in combustion chamber pressure or throat area, and this added weight required would probably result in a final net saving in gross weight which is practically zero. Also, such a system would present a much more complex engineering problem. Hence since the saving in gross weight, if any, is small and the increase in complexity is large, the succeeding analysis will be based entirely on a burning system with constant rate of mass flow of propellants. However, the whole question of the staging of n and the method of burning should be further investigated if small gains are desired.

10. Considerations Concerning the Optimum Number of Stages

The discussion so far has resulted in simplifications in the analysis to the extent of using constant ν, constant n, and constant rate of mass flow of propellants. Since these discussions were based on a non-variable number of stages N, the optimum number of stages which the rocket should have may now be considered. The performance equation (30) may be written in the form

$$N g_s I \log \frac{1}{1-\nu} = \Delta v + \sum_{j=1}^{N} \overline{g_j \sin \theta_j} \times t_{b_j} \quad . \tag{37}$$

As usual, Δv is considered as having a constant fixed value. If, in addition, it is assumed that the discussion is based on trajectories which are more or less geometrically similar, then the term

$$\sum_{j=1}^{N} \overline{g_j \sin \theta_j} \times t_{b_j}$$

will remain approximately constant, and thus the right hand side of Eq. (37) may be considered to remain constant. Eq. (37) then shows, in general, that when N increases, ν must decrease. Assuming for simplicity that $(W_B/W)_j = (W_B/W)_1 =$ const. for any given trajectory, it then follows from Eq. (25) that an optimum value must exist for the number of stages N. Using appropriate trajectories and the smallest values of $(W_B/W)_i$ consistent with present day materials and structural techniques, it is found that the optimum N has a value of 2, 3, or 4 depending upon the propellants (through the specific impulse I which are used. Hence most of the discussion, analysis, and calculations contained in the remainder of the report will be restricted to a rocket having either two or three stages. The four stage rocket is not given any further consideration because the gains are very small and the increase in complexity is considerable.

11. Drag

In the simplified performance equation (30) certain quantities and refinements were neglected in order to simplify the discussion, and these will now be taken into account. First the lift and drag of the rocket, which were neglected in Eq. (30), must be considered. Of these two forces it is only the drag D of the rocket which enters the equation of motion (6) along the path, and this becomes

$$\frac{dv}{dt} = \frac{F}{m} - g \sin\theta - \frac{D}{m} \quad . \tag{38}$$

When a lift force L normal to the trajectory is present (lift positive in the direction of $\theta = +\pi/2$, the equation of motion (6) becomes

$$\frac{d\theta}{dt} = \frac{1}{v}\left(\frac{L}{m} - g\cos\theta + \frac{v^2}{r}\cos\theta\right) \quad , \tag{39}$$

which may be approximated by the difference equation

$$\Delta\theta = \frac{1}{v}\left(g_s\frac{L}{W} - g\cos\theta + \frac{v^2}{r}\cos\theta\right)\Delta t \quad . \tag{40}$$

At this point we may also take account of the fact that the specific impulse I will vary slightly with staging, owing mainly to the changes in the free-air pressure, so that the performance equation is now given by the expression

$$\Delta v = \sum_{j=1}^{N}\Delta v_i = g_s\sum_{j=1}^{N}\overline{I}_j\log\frac{1}{1-\nu} - \sum_{j=1}^{N}\overline{g_j\sin\theta_i}\times t_{b_j}$$

$$- g_s\sum_{j=1}^{N}\left(\frac{\overline{D}}{W}\right)_j \times t_{b_j} \quad , \tag{41}$$

where the bar indicates the mean value[7] during any particular stage, and where $W = mg_s$ is the weight referred to standard sea-level conditions, g_s denoting standard gravity at sea level (g_s = 32.174 ft/sec^2).

Since the drag is discussed in detail in Ref. (6), it will only be mentioned here that as far as practical trajectories of interest are concerned, the maximum value of the ratio D/W in the first stage is about 0.5, and its average value during this stage is about 0.2. These values become considerably smaller in the succeeding stages where the drag approaches zero owing to high altitude. Thus, since $\overline{(D/W)}_j$ is relatively small compared to $\overline{\sin\theta_j}$, the previous arguments based on the simplified performance equation (30) are not invalidated by the neglect of the drag term. In view of the more exact performance equation, (38), it follows that the maximum load factor n is no longer given by Eq. (14), but must now be defined by

$$n = \frac{1}{g_s}\left(\frac{F-D}{m}\right)_{max} = \left(\frac{F-D}{W}\right)_{max} , \tag{42}$$

where W is the weight referred to standard sea-level conditions. It was pointed out in connection with Eq. (17) that when the burning occurs with constant rate of mass flow of propellants, the maximum value of F/W occurs at the end of a burning period. On the other hand, the ratio D/W becomes quite small at the end of a burning period, typical values being 0.05 at the end of the first period and 0.00 at the end of the second period. Accordingly, here again, the inclusion of the drag term has little influence, and the maximum load factor is still determined for all practical purposes by $(F/W)_{max}$ as before.

12. Lift, Guidance, and Tilting

As far as lift (force normal to trajectory) is concerned, this is required in order to achieve the major control over the shape of the trajectory, particularly in order to turn the trajectory into the circular orbit, and for these purposes the lift required will always be directed toward the center of curvature of the trajectory and will therefore be negative. The need for lifting forces will become apparent from the following remarks. First of all it should be pointed out that because of the high temperatures existing in the ionosphere(4), the decrease in density, and therefore drag, with height is relatively small, making it necessary to establish the orbit at a height of 300 to 350 miles if the endurance is to be of the order of several months. In investigating possible trajectories it is necessary to consider the fact that the air resistance, starting from its initial value of zero, will increase rapidly at first as the rocket accelerates in the high density portion of the atmosphere. As the rocket gains altitude and the density becomes small, the drag forces begin to decrease and become negligible at a height of about 250,000 ft. The variation of rocket speed with height and of atmospheric density with height are such that the maximum drag

[7] In view of the convention adopted in footnote [6], the drag term would be written $\overline{(D_{tj}/W_{tj})} \equiv \overline{(D_t/W_t)}_j$ to indicate that the mean value is based on values of drag and weight which are variable with time. However, since this is clear from the context, the subscript t will be omitted

occurs at a height of the order of 50,000 ft. It is apparent that the initial portion of the trajectory should be nearly vertical so as to reduce as much as possible the length of flight path over which the rocket is subject to appreciable drag. On the other hand, if too much of the trajectory is steep (large sin θ) there will be an adverse effect on velocity because of the presence of the gravity term $\overline{g_j \sin \theta_j} \times t_{b_j}$ in the performance equation (41). Thus to minimize the adverse effects of the drag and gravity terms in Eq. (41) it is necessary to arrange that the rocket travel over a specified trajectory, which can be achieved only by the use of lift forces.

Although some negative lift could be obtained from the body and external surfaces of the rocket by operating the rocket at an angle of attack, these, in general, are not nearly sufficient to control the trajectory, especially in the first stage where relatively large amounts of lift are required and where aerodynamic lift is small owing to the relatively low speed. Moreover, at the heights corresponding to the upper part of the trajectory the density of the air becomes so low that, in spite of the great speed, the dynamic pressures are inadequate to produce any appreciable aerodynamic lift. Accordingly, the only practical scheme for producing lift for trajectory control must be based on the use of rocket motor thrust. The question then arises whether it is better to use a number of small rocket motors thrusting in a direction normal to the centerline of the rocket and passing through the center of gravity, or to have an arrangement using the moment produced by small rocket motors to tilt the rocket and thereby give a component of the thrust of the main rocket motor in a direction normal to the trajectory (lift) which may be used to provide the necessary lift. These two methods of obtaining the necessary lift for trajectory control, or guidance, are illustrated in Fig. 4. It is readily shown that as far as propellant consumption is concerned, the latter arrangement is much superior. Although the basic argument used to show this has been presented previously in the first satellite investigation(1), it is included here, since it is essential for completeness and continuity, and since it also provides the opportunity for making necessary changes in notation.

Let L denote the lift force normal to the trajectory, positive in the direction $\alpha = +\pi/2$, and let α be the angle of attack (angle of tilt of rocket). The angle of attack α is defined as the acute angle measured from the tangent to the trajectory (the direction of air flow) to the centerline (zero-lift chord line) of the rocket. The angle α is taken as positive in the direction opposite to that of the center of curvature of the trajectory, that is positive in the $+\theta$ direction.

Case 1. Lift Obtained from Small Rocket Motors

It will be assumed that the specific impulse of the small rocket motors is the same as that of the main motor in the satellite rocket. Using the thrust expression (7), the thrust F_0 required from the main motor when none of this thrust is used for lift may be written

$$F_0 = -g_s \left(\frac{dm}{dt}\right)_0 I \quad ,$$

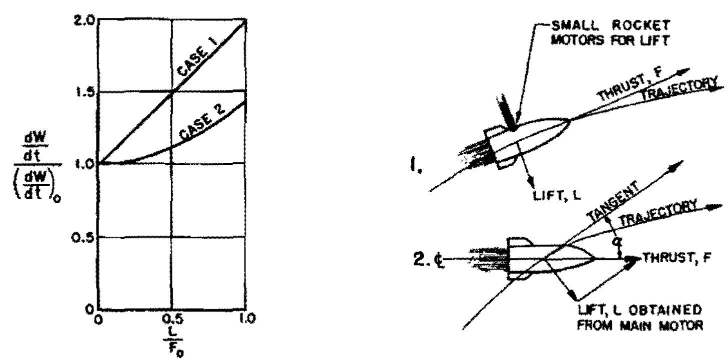
where $(dm/dt)_0$ is the mass rate of propellant consumption corresponding to the thrust F_0. The lift force produced by the small rocket motors is

$$L = -g_s \left(\frac{dm}{dt}\right)_1 I \quad ,$$

where $(dm/dt)_1$ is the propellant consumption required by the lift motors. Since, with this arrangement all of the main rocket motor thrust is directed along the trajectory, the thrust remains at the value F_0. Hence the ratio of the mass rate of propellant consumption with and without guidance for Case 1 is given by

$$\frac{\dfrac{dm}{dt}}{\left(\dfrac{dm}{dt}\right)_0} = \frac{\left(\dfrac{dm}{dt}\right)_0 + \left(\dfrac{dm}{dt}\right)_1}{\left(\dfrac{dm}{dt}\right)_0} = 1 + \frac{\left(\dfrac{dm}{dt}\right)_1}{\left(\dfrac{dm}{dt}\right)_0} = 1 + \frac{L}{F_0} \,, \tag{43}$$

where dm/dt is the total rate of propellant consumption.



F_0 = axial thrust required
L = lift required
$\frac{dW}{dt}$ = weight rate of propellant flow
$\left(\frac{dW}{dt}\right)_0$ = weight rate of propellant flow corresponding to all thrust in the axial direction and equal to F_0.

Case 1.
lift obtained by using small rocket motors.

Case 2.
lift obtained by using a component of main rocket motor by tilting the rocket.

PLOT SHOWING SUPERIORITY OF OBTAINING LIFT BY USING A COMPONENT OF THE MAIN ROCKET MOTOR THRUST

FIG. 4

Case 2. Lift Obtained as a Component of the Main Rocket Motor Thrust

In this case, Fig. 4, since the vehicle must be tilted with respect to the trajectory, the thrust required of the main rocket motor must be sufficient to produce a thrust F_0 directed along the trajectory and a force L normal to the trajectory. Denoting by F_2 the main rocket motor thrust required in this case, we have

$$F_0 = F_2 \cos\alpha = -g_s \left(\frac{dm}{dt}\right)_2 I \cos\alpha \,, \text{ and}$$

$$L = F_2 \sin\alpha = -g_s \left(\frac{dm}{dt}\right)_2 I \sin\alpha \,,$$

where $(dm/dt)_2$ is the total weight rate of propellant consumption, and where $F_2 = \pm\sqrt{F_0^2 + L^2} = -g_s I\,(dm/dt)_2$. Therefore the ratio of the propellant consumption rates for Case 2 is

$$\frac{\dfrac{dm}{dt}}{\left(\dfrac{dm}{dt}\right)_0} = \frac{\left(\dfrac{dm}{dt}\right)_2}{\left(\dfrac{dm}{dt}\right)_0} = \sqrt{1 + \left(\frac{L}{F_0}\right)^2} \quad . \tag{44}$$

Having the ratios (43) and (44), one may compare the relative rate of propellant consumption corresponding to the two methods for achieving lift.

The ratio of the consumption rates with and without small auxiliary rocket motors are plotted in Fig. 4 against the ratio L/F_0. It is immediately apparent that Case 2 is markedly superior to Case 1. In fact, using the scheme of Case 2, substantial guidance forces may be obtained without appreciable penalty in thrust, and the method of Case 2 will therefore be adopted as the means whereby the lift forces are obtained. Since jet vanes are not to be used, this method of obtaining lift will require the use of certain rocket control motors to produce the necessary tilt of the rocket. These rocket motors are to be movable to the extent that they can produce thrusting in various directions and in this way produce moments about a transverse axis. It may be shown, by an argument similar to that used above, that the total propellant consumption of the control rocket motors and the main rocket motor will be least when the control motors are as large as possible. This means that the main rocket motor should actually consist of a group of movable control motors which are used not only to provide the necessary thrust for propulsion but also to provide the necessary control moments(13).

Since the scheme adopted for achieving lift requires that the rocket have a variable angle of tilt α, this must now be taken into account by the introduction of the factor $\cos\alpha$ in the performance equation, since the thrust used in that equation is always defined as the thrust directed along the trajectory. Hence, with angle of tilt present, the component of thrust along the trajectory is, from Eq. (7),

$$F\cos\alpha = -g_s \frac{dm}{dt} I \cos\alpha \ ,$$

where F is the total thrust of the rocket motor. Using this expression for the thrust occurring in Eq. (38), the performance equation becomes

$$\Delta v = g_s \sum_{j=1}^{N} \overline{(I\cos\alpha)_j}\, \log\frac{1}{1-\nu} - \sum_{j=1}^{N} \overline{g_j \sin\theta_j} \times t_{b_j}$$

$$- g_s \sum_{j=1}^{N} \left(\frac{\overline{D}}{W}\right)_j \times t_{b_j} \quad . \tag{45}$$

Similarly the expression (42) for the maximum load factor must now be written

$$n = \frac{1}{g_s}\left(\frac{F}{m}\cos\alpha - \frac{D}{m}\right)_{max} . \tag{46}$$

However, preliminary calculations based on actual typical flight conditions and trajectories have shown definitely that the effect of the factor cos α and of the term $(D/m)_{max}$ is entirely negligible so far as Eq. (46) is concerned; and it is quite adequate as far as structural load is concerned to use the simpler expression, Eq. (17),

$$n = \frac{\nu\ (I_j)_{max}}{(1-\nu)\ t_{b_j}} , \tag{47}$$

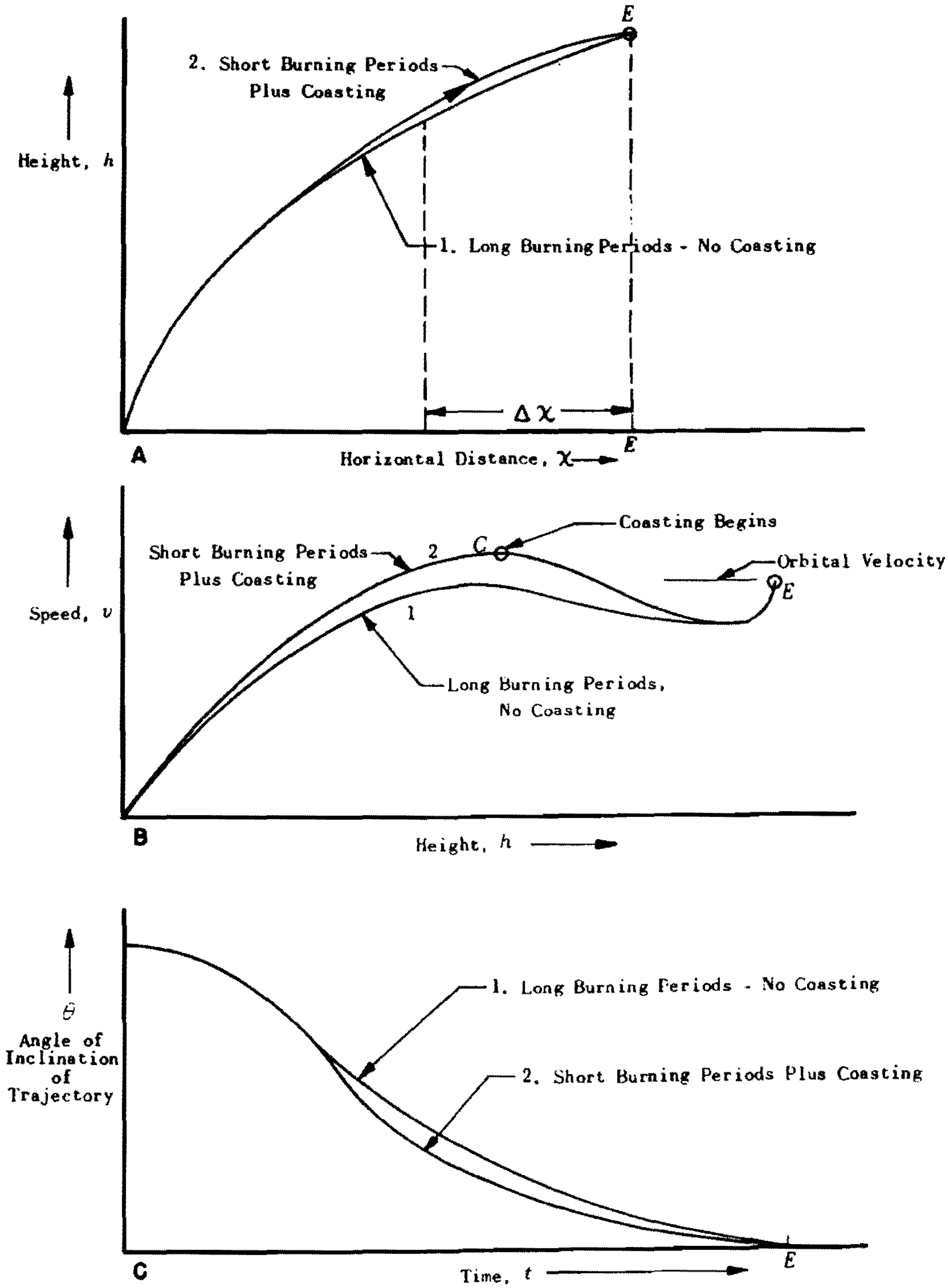
where it is to be so arranged, as discussed previously in connection with the optimum value of n, that the maximum load factor n has the same value in all stages, since this represents the best that can be done as far as staging of n is concerned. The quantity $(I_j)_{max}$ represents the maximum value of the specific impulse in any particular stage which, owing to the beneficial influence of decreased external pressure, will always occur at the end of a stage. Since the rocket is to be operated with n and ν constant, it is seen from Eq. (47) that this corresponds to the condition that $(I_j)_{max}/t_{b_j}$ remain constant with staging over any given trajectory. This may be readily achieved by the proper choice of motor size (see Refs. 6 and 14).

In determining the best trajectory, it must be decided how the tilting, and therefore the lift forces, must be programmed. As already pointed out, this must not only be such as to minimize the adverse effects of the drag and gravity terms in the performance equation (45), but also such that the angle of tilt necessary to bend the trajectory does not become so large that excessive decreases result in the component of thrust along the trajectory, since this is the effective thrust producing the acceleration. In view of the relatively great height at which the orbit must be established, it is found that these conditions can best be met by the proper use of a period of coasting along the trajectory.

Actual trajectory calculations have shown that it is best to launch the rocket in the direction of the vertical and to have most of the tilt very near the beginning of the flight where its bending effect on the path is largest and where the maximum angle of tilt is only a few degrees. It might appear that non-vertical launching would be better inasmuch as this would require less tilting. However, this method of launching turns out to be inferior because it results in a greater length of flight path in the high drag region of the atmosphere. Moreover, it is highly undesirable because a launching track would be necessary.

13. Coasting

Although coasting motion was not explicitly involved in the foregoing discussions of the performance equation, it was implied nevertheless by the relation $V_{orb} = \Delta v + \Delta v_c$ (mentioned in section 6) that such motion might be present. In order to establish efficiently an orbit at high altitudes (it appears that the height of the



SCHEMATIC DIAGRAM TO ILLUSTRATE COASTING

FIG. 5

orbit must be about 350 miles), it turns out to be highly desirable to have a period of coasting (non-powered flight) somewhere near the end of the trajectory. The reason for this will become apparent from the following qualitative discussion.

The basic requirement of the trajectory is that it must lead to final values of speed (v_F) and direction ($\theta_F = 0$) appropriate for the establishment of a circular orbit situated at a sufficiently great height (about 350 miles) to satisfy endurance requirements. The height attained by the rocket is determined by the integral

$$h = \int_0^T v \sin \theta \, dt \tag{48}$$

where T is the total duration of the flight over the trajectory. Neglecting for the moment any variation in T, it is apparent from this equation that the greatest height is attained when v and θ are so scheduled that θ is large when v is small, and conversely, v is large when θ is small. Considering the time T, it is also clear that when the scheduling of v versus θ is fixed, h will increase with increasing T so long as θ remains positive. A large value of T may be obtained either by employing long burning periods, which by Eq. (31) corresponds to small n, or by employing shorter burning periods but allowing the rocket to travel over part of the trajectory without power, that is, by allowing the rocket to coast. The question then arises as to which is the better method to use in order to attain the high altitude necessary for the orbit.

Consider two trajectories, one in which either one or more long burning periods are used and the other in which shorter burning periods and coasting are used. Suppose the tilting program is so arranged that the two trajectories have the same final orbital conditions, curves A, Fig. 5. Depending upon the length of the burning periods, the rocket will require different amounts of time to reach the same height but will always have the same speed (orbital velocity) at the end of the trajectory, point E. If short burning periods and coasting are employed (curves B) with coasting occurring near the end of the trajectory (it will be shown that coasting is best as near the end as possible), the velocity may at some height, point C, be greater than the orbital velocity, since the vehicle must slow down in the coasting region C to E. When there is no coasting and all of the trajectory is under powered flight with long burning periods, the speed of the rocket in the latter part of flight decreases less with height than when coasting is present, finally attaining the orbital velocity at E, curve 1 of Fig. 5, B. Accordingly, for given heights along the trajectory particularly in the latter part except the end points, the velocity for the case with short burning periods and coasting will, in general, be greater than for the case with long burning periods as indicated in B of Fig. 5. The rapid change in velocity with altitude indicated at the very end of curves B is due to the necessary final boost in speed required to attain orbital conditions after coasting or when slow burning is employed. This matter is discussed further later on.

According to Eq. (40) when the lift is negligibly small and v^2/r is $< g$, for a given distance Δx back from the end of the trajectory, the change in θ is less for the case with coasting than for the long burning period case. Hence, in A of Fig. 5

the trajectory with coasting is the higher of the two. By consideration of the curves of A and B and the expression

$$T = \int_0^E \frac{dx}{v \cos\theta} \quad \text{or} \quad T = \int_0^E \frac{dh}{v \sin\theta}$$

for the total time of flight to traverse the trajectory, it is seen that the flight time for the trajectory with coasting is less than or at most equal to the flight time for the long burning period case. It also follows from Eq. (40) that the change in θ, proceeding backward from E, will be indicated in C of Fig. 5, that is, smaller for the coasting case. It may be noted that in practice the curves of B and C may have several slope discontinuities.

It has been pointed out previously that the final velocity attained by the rocket when coasting is present is given by $v_F = \Delta v + \Delta v_c$, where Δv_c is the velocity change during coasting. The term Δv_c is entirely similar in character and effect to the gravity terms in $g_j \sin\theta_j$ which occur in the performance equation. The entire effect of gravity upon trajectory performance could therefore be summed up by an expression of the form

$$\int_0^T g \sin\theta \, dt ,$$

where T is the total time of flight over the trajectory including coasting. From a consideration of C of Fig. 5 it follows that this integral is less for the case with coasting present. Since from trajectory considerations it is desirable to have gravity effects as small as possible, it is seen that this may be accomplished by using coasting (near the end of the trajectory) and burning periods as short as possible. Although the discussion of coasting thus far has been very qualitative in nature, it has been found justified as the result of some earlier trajectory studies. In fact these results are more generally true than here discussed; for example, the simplification of having the same range need not have been made. The result is true even when the cases are compared at their optimum ranges.

As far as the effect of coasting on the determination of n is concerned, if the discussion of n in section 9 were now repeated to include coasting, the conclusions reached before would remain unchanged. This follows because when altitude is obtained by coasting, it is still desirable from trajectory considerations to have the maximum load factors as large as possible and since this was also the basis of the arguments in section 9, the conclusions reached there remain unchanged. Likewise, the effect of introducing a period of coasting in the performance equation is negligible as far as the other previous results and conclusions associated with this equation are concerned.

Having pointed out the desirability of attaining altitude by the use of coasting with v large when θ is small, the question which then arises is how best to gain altitude in the lower part of the trajectory where v is relatively small and where $\theta \doteq 90°$. This might be done, for example, by launching the rocket vertically ($\theta \doteq 90°$) and then starting the tilt late but using large amounts of tilt to bend the path sufficiently rapid to give the required condition $\theta_F = 0°$ at the end of the trajectory. However, this would be an inefficient method of obtaining altitude since it would result in a large reduction in velocity due to the effect of gravity ($g \sin \theta$) and a large reduction in thrust due to the large tilt angles ($F \cos \alpha$). Although it is always best to have launching in the vertical, the trajectory studies show that the most efficient tilt program is such that the maximum is small (never more than a few degrees) and occurs early in the first stage, while in the succeeding stages the tilt is zero.

Since it is advantageous to have a period of coasting in the trajectory, a coasting term must be introduced into the performance equation. Since the coasting period will always be located near the end of the trajectory where drag is negligible, the rocket may be considered as a free body moving in a potential (conservative) force field, the field of gravity. Consequently, along the coasting part of the trajectory, the motion of the rocket will be governed by the condition that the total energy (potential plus kinetic) remain constant, which is expressed by the equation

$$\frac{1}{2} m v_i^2 - \frac{1}{2} m v_f^2 = m \int_{h_i}^{h_f} g dh \, , \tag{49}$$

where the mass of the rocket will be constant since there is no consumption of propellants, and where h is height measured in the vertical. The subscript i denotes the beginning of the coasting period and the subscript f the end of the period. Replacing the g under the integral by a mean value $\overline{g}_c$ appropriate to the height interval $h_f - h_i$, this may be written

$$v_i^2 - v_f^2 = \overline{2g_c} \Delta h_c \quad , \tag{50}$$

where $\Delta h_c = h_f - h_i$ is the change in height during coasting. From this, it follows immediately that

$$\Delta v_c \equiv v_f - v_i = -\frac{\overline{2g_c} \; \Delta h_c}{v_i + v_f} \quad , \tag{51}$$

where Δv_c is the decrease in velocity during the coasting period. Since the coasting will always occur near the end of the trajectory where v_i and v_f do not differ greatly it is sufficiently accurate to write

$$\Delta v_c = -\frac{\overline{g}_c \Delta h_c}{\overline{v}_c} \quad , \tag{52}$$

where v_c is the mean velocity during coasting.

In the performance equation (45) the velocity change Δv refers only to the change velocity which results while the rocket motors are in operation. By introducing the expression (52) for the velocity change during coasting, Eq. (45) becomes

$$\Delta v_T = g_s \sum_{j=1}^{N} \overline{(I \cos \alpha)}_j \log \frac{1}{1-\nu} - \sum_{j=1}^{N} \overline{g_j \sin \theta_j} \times t_{b_j}$$

$$- \sum_{j=1}^{N} g_s \overline{\left(\frac{D}{W}\right)}_j \times t_{b_j} - \frac{\overline{g}_c \Delta h_c}{\overline{v}_c} , \tag{53}$$

where Δv_T is the total velocity change resulting from rocket motor operation and coasting. Since α is always very small, never exceeding several degrees, $\cos \alpha$ may be replaced by unity as far as Eq. (53) is concerned. Thus, when the vehicle starts from rest under its own power and finally attains the required orbital velocity v_{orb}, we have $\Delta v_T = v_{orb}$ and, if t_{b_j} be replaced in terms of Eq. (31), the performance equation (53) may then be written

$$g_s \sum_{j=1}^{N} \overline{I}_j \log \frac{1}{1-\nu} = v_{orb} + \sum_{j=1}^{N} \frac{\nu}{(1-\nu)n} \overline{g_j I_j \sin \theta_j}$$

$$+ \sum_{j=1}^{N} \frac{g_s \nu \overline{I}_j}{(1-\nu)n} \overline{\left(\frac{D}{W}\right)}_j + \frac{\overline{g}_c \Delta h_c}{\overline{v}_c} . \tag{54}$$

Suppose that the powered flight is interrupted somewhere in the trajectory by a period of coasting and that the orbital velocity to be attained is specified and therefore a constant. Relatively speaking, as coasting occurs successively later in the trajectory, v_i becomes larger, which causes $\overline{v}_c$ to become larger, which in turn causes $\overline{g}_c \Delta h_c / \overline{v}_c$ to become smaller. Since for approximate results, it is permissible to treat $\overline{I}_j$ and $\sum_{j=1}^{N} \overline{\left(\frac{D}{W}\right)}_j$ as quantities which remain constant, it then follows from a consideration of Eq. (54) that the value of ν must also decrease as v_i (or $\overline{v}_c$) increases. Since, as pointed out previously, a small value of ν leads to a small value for the initial gross weight W_1 we always seek conditions leading to small ν. Therefore, in view of the coasting effect on ν, this alone would indicate that the coasting should take place as near as possible to the end of the trajectory where $\overline{v}_c$ has its greatest value.

On the other hand, as the coasting occurs successively later in the trajectory, the average value of $\sin \theta$ over the entire trajectory will increase, since the greater length of trajectory with powered flight before coasting results in greater length of trajectory with the larger values of θ. A systematic investigation, using accurate trajectory calculations, has shown that the effect of the positioning of coasting on ν is more important than the effect on θ, and that in order to obtain least ν, the quantity $\overline{v}_c$ should be as large as possible, a condition which is satisfied when the coasting is positioned as near to the end of the trajectory as possible. In this connection however, it must be pointed out that at the very end portion of the trajectory it is necessary to have a final burst (short duration) of powered flight to secure the exact speed and direction which the vehicle must have in order to establish the circular orbit. Thus the period of coasting may not extend completely to the final end point of the trajectory where the circular orbit is established. Taking this restriction into account, detailed trajectory studies reveal that the optimum position for coasting is within the last burning period (i.e. within the last stage) beginning at that point on the trajectory where the last period of burning has reached about 97 per cent completion. The motor is shut off at this point allowing a specified period of coasting. The motor is then turned on again for the final burst of powered flight which has a duration equal to the remaining 3 per cent of the last burning period.

Let τ denote the percentage completion of the last burning period when coasting begins. Then, consistent with the restrictions imposed by the necessity of having a final burst of powered flight, the optimum coasting occurs when $\tau \doteq 0.97$. If the coasting is positioned farther than this along on the trajectory, it becomes impossible to establish the conditions for a circular orbit. Furthermore, positioning the coasting at the value $\tau = \tau_{max} \doteq 0.97$ as found above, results in such a long range (practically half way around the earth, which is too far from the standpoint of communication limitations) before the circular orbit is established that it is found necessary, on this account, to shorten the range by using τ approximately equal to 0.75 and having the remaining 25 per cent of the last burning period occur after coasting. Using the value $\tau = 0.75$ as demanded by range limitations, it is found that the gross weight is about 15 per cent greater than if the optimum position of coasting were used. The systematic investigation of this will be discussed in Part II.

Accordingly, since the coasting occurs so late in the trajectory, the mean speed $\bar{v}_c$ will not differ much from the orbital speed v_{orb} and it is sufficiently accurate to use $\bar{v}_c = v_{orb}$ in Eq. (54) which may then be written

$$\sum_{j=1}^{N} \bar{I}_j \log \frac{1}{1-\nu} = \frac{v_{orb}}{g_s} + \sum_{j=1}^{N} \frac{\nu}{(1-\nu)n} \overline{\frac{g_j}{g_s} I_j \sin\theta_j} + \sum_{j=1}^{N} \frac{\nu}{(1-\nu)n} \bar{I}_j \left(\overline{\frac{D}{W}}\right)_j + \frac{\bar{g}_c}{g_s}\frac{\Delta h_c}{v_{orb}} \ . \tag{55}$$

This simplified approximate equation is of fundamental importance in the analysis of the rocket and its motion since it expresses, in a highly practical form, the basic relationship existing between the fundamental parameters.

The performance equation (55) finds its principal application when it is desired to find the effect on ν of changes in the other parameters. For example, if the main features of the tilt α and coasting τ programs are kept fixed so that the main features of the shape of the trajectory remain the same, the formula (55) may be used to great advantage for deriving the change in ν resulting from variations in the values of the other variables such as I, h_{orb} (orbital height), and N. One use of

Eq. (54) should be mentioned in particular; since it is best to have the coasting occur very near the end of the trajectory, the positioning of some of the coasting earlier in the trajectory will be inefficient. However, some coasting of this nature is unavoidable owing to the small time interval which elapses between stages. Calculations for a three stage rocket of the increase in ν caused by three seconds coasting between the first and second, and between the second and third burning periods, resulted in a 2 per cent increase in gross weight.

The formula (55) may also be used to estimate the change in ν with height, and for this purpose it is convenient to make a further modification. First, it may be pointed out that the minimum gross weight studies have shown that the optimum values of n must lie within the range $4 < n < 8$. It is found from many trajectory calculations that when the orbital velocity is attained under optimum conditions, the burning times are such that the burning is practically completed by the time the vehicle has attained a height of the order of 100 miles, that is, the non-coasting change in altitude is about 100 miles. The condition for the circular orbit is that the centrifugal force be equal to the gravitational force, which is expressed by

$$\frac{v_{orb}^2}{r} = g = g_R\left(\frac{R}{r}\right)^2 , \tag{56}$$

where r is the distance of the orbit from the center of the earth, R is the radius of the earth, and g_R is the absolute value of gravity at sea level. The orbital velocity is therefore given by

$$v_{orb} = \sqrt{g_R R^2} \times \frac{1}{\sqrt{r}} = \sqrt{g_R R^2} \times \frac{1}{\sqrt{R+h}} = \sqrt{g_R R} \times \left(1 + \frac{h}{R}\right)^{-\frac{1}{2}} , \tag{57}$$

where $h = r - R$ is the height above sea level. Since h/R is small compared to 1, the last expression on the right may be expanded by the binomial theorem, and if only the first two terms in the expansion are retained, the orbital velocity may be expressed to a high degree of approximation by[8]

$$v_{orb} = \sqrt{g_R R}\left(1 - \frac{1}{2}\frac{h}{R}\right) . \tag{58}$$

Since the orbit is to be established at a height of the order of 300 miles and since powered flight ends at about 125 miles height, the heights of interest for coasting motion will range from 100 to 400 miles. Let v_{orb_2} and v_{orb_1} be the orbital velocities at the heights h_2 and h_1 respectively, where $h_2 > h_1$. If B' denotes the rate of change of orbital velocity with height, it follows from Eq. (58) that

$$B' \equiv \frac{v_{orb_2} - v_{orb_1}}{h_2 - h_1} = -\sqrt{\frac{g_R}{4R}} . \tag{59}$$

[8] In this last expression only the first two terms are retained as an approximation to the expansion of $(1 + h/R)^{-1/2}$. For values of h of interest here, h/R is of the order of 0.1 and the approximation is accurate within 1 per cent which is entirely adequate for the present analysis.

The orbital velocity at a height h, where $h_1 < h < h_2$, may be expressed in terms of v_{orb_2} by the relation

$$v_{orb} = v_{orb_2} - B' (h_2 - h) ,$$

which may also be written in the form

$$v_{orb} = v_{orb_2} \left[1 + \frac{B(h_2 - h)}{100} \right], \tag{60}$$

where B is a constant having the value $B = + \dfrac{5.280 \times 10^5}{v_{orb_2}} \sqrt{\dfrac{g_R}{4R}}$ and where h is expressed in miles. This enables the height interval $(h_2 - h)$ to be expressed in terms of a unit of distance of 100 miles, which has certain advantages as far as the calculations are concerned. Using the relation (60) with h_2 = 300 miles, $v_{orb_2} = v_{orb_{300}}$, using $\Delta h_c = h - 125$ since coasting begins at about 125 miles altitude, and neglecting the minor effects due to variations of gravity, the performance equation (55) is finally written in the simplified form

$$\sum_{j=1}^{N} \overline{I_j} \log \frac{1}{1-\nu} = \frac{v_{orb_{300}}}{g_s} \left[1 + \frac{B(300 - h)}{100} \right] + \sum_{j=1}^{N} \frac{\nu}{(1-\nu)n} \overline{I_j \sin \theta_j}$$

$$+ \frac{\nu}{(1-\nu)n} \overline{I_1} \left(\frac{\overline{D}}{W} \right)_1 + \frac{5.280 \times 10^5}{v_{orb_{300}}} \left(\frac{h - 125}{100} \right) \tag{61}$$

in which the height h occurs explicitly, and where the quantity $\left[1 + B \left(\dfrac{300 - h}{100} \right) \right]$ which strictly should also appear in the denominator of the last term has been replaced by unity without affecting the accuracy of the equation. The constants have been based upon a height of 300 miles since this is approximately the required orbital height for the satellite rocket. From this equation it is found, for example, by using constants corresponding to a three stage hydrazine-oxygen trajectory, and by neglecting the variation of all terms on the right side of Eq. (61) not containing h explicitly, that $d\nu/dh_{orb}$ is such that the gross weight changes about 10 per cent with a 100 mile change in orbital altitude.

The simplified performance equation (61) is very useful. For example, it may be used to advantage in obtaining an initial estimate for the value of ν to be used in the exact trajectory calculations. Its most important use, as pointed out in connection with Eq. (55), is in determining the changes in ν which result when the parameters I, n, h, N, and $\overline{\sin \theta_j}$ are varied one at a time. The most important of these is the dependency of ν on I. This effect may be studied for changes in I up to 15 per cent by holding constant the right hand side of Eq. (61). It is also found, from accurate trajectory calculations, that the right hand side of Eq. (61) may be considered to remain constant in determining the change in ν due to change in N, provided N does not change by more than ±1. The variation of the initial gross weight W_1 with the number of stages N based on the final design value of ν for the three stage hydrazine-oxygen rocket is shown in Fig. 18.

The simplified formula Eq. (61) is not suitable for revealing the change in ν resulting from change in n because n is strongly involved implicitly in the term $\Sigma \overline{\sin \theta_j}$.

The use of Eq. (61) for computing change in ν due to change in $(\overline{D/W})_j$ gives results of quite satisfactory accuracy. This enables one to ascertain the change in ν shape of the rocket and is therefore important in determining the optimum shape for least gross weight as this depends on drag as well as structural weight. This will be discussed further in Ref. (6).

Another interesting use of formula, Eq. (61), is the calculation of the reduction in ν which results when the rocket is launched from a high altitude such as a mountain top. If the rocket were launched at a height of 10,000 feet, for example, the reduction in drag due to the lower density at launching results in a decrease of 3 per cent in the gross weight.

II. FLIGHT MECHANICS AND TRAJECTORY CALCULATIONS

In part I the problems of establishing a rocket of minimum gross weight on a circular orbit as a satellite of the earth have been discussed from a more or less general point of view and in a somewhat qualitative fashion. By means of this somewhat elementary but fundamental discussion, it was possible to derive a simplified performance equation, (61), which is extremely useful in showing how the various performance parameters are related. These approximate relations must now be treated on the basis of more exact mathematical formulations which will lead to accurate determinations of trajectories and of the optimum values for the various performance and trajectory parameters.

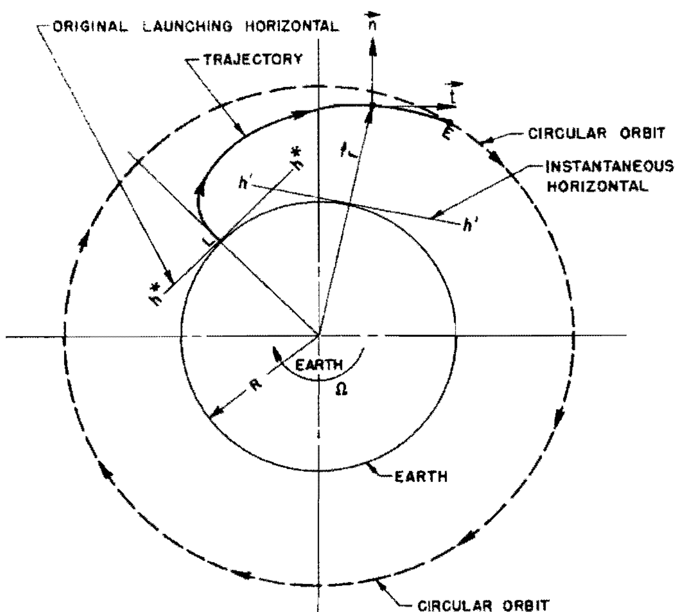
1. The General Equations of Motion of a Point Mass Moving on a Path in the Equatorial Plane of a Rotating Planet.

Since the present study is concerned only with the special case in which the satellite orbit lies in the equatorial plane, the analysis of the motion will become somewhat simpler. Although the equations of motion may be derived with neatness and brevity by employing Lagrange's dynamical equations, a longer but perhaps more physically significant method will be used here. The basic equations of motion are those for a point mass moving on a path situated in the equatorial plane of the rotating earth. Consider a rectangular system of coordinates having its origin at the center of the earth and rotating with the earth at the constant angular velocity $\vec{\Omega}$, where the arrow is used to denote a vector quantity. The vector $\vec{\Omega}$ passes through the center of the earth and is directed toward North. If $\vec{f}$ is the true resultant force acting on the particle of mass m, the vector force equation is[5]

$$\frac{\vec{f}}{m} = \vec{a}' + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) + 2\,\vec{\Omega} \times \vec{v}' , \tag{62}$$

where $\vec{r}$ is the position vector of the particle measured from the origin at the center of the earth, $\vec{a}'$ is the apparent linear acceleration of the particle, and $\vec{v}'$ is the apparent velocity of the particle, $\vec{r}$, $\vec{a}'$, and $\vec{v}'$ all being measured relative to the rotating system of coordinates. The term $\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$ represents the centripetal reaction, and $2\vec{\Omega} \times \vec{v}'$ is known as the Coriolis' acceleration. The position vector $\vec{r}$ has the same magnitude regardless of whether the system of reference is rotating or not.

Since the motion of the point mass is to be identified with the motion of the rocket, we consider a trajectory, as shown in Fig. 6, which starts at the earth's surface at the launching point L, travels in a clockwise (eastward) direction of rotation with respect to the earth, and establishes the circular orbit at the end of the trajectory at the point E.



SCHEMATIC DIAGRAM OF TRAJECTORY AND ORBIT OF ROCKET IN THE EQUATORIAL PLANE AS VIEWED FROM THE SOUTH

FIG. 6

The earth rotates in a clockwise direction (when viewed from the South) with angular velocity Ω. The line h^*h^*, which may be referred to as the original horizontal, specifies the direction of the horizontal at the time of launching and rotates with the earth in a clockwise direction. Only the clockwise (eastward) direction of motion of the vehicle will be considered here, since this case requires less energy to establish the orbit than when the motion is in the counterclockwise direction. Since the position vector is restricted to the plane of the equator and since the angular velocity vector $\vec{\Omega}$ is normal to the equatorial plane, and points North, the vector product $\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$ will be a vector in the direction of $-\vec{r}$, and the vector product $\vec{\Omega} \times \vec{v}\,'$ will be a vector normal to the path. With the type of trajectory being treated here, Fig. 6, it is apparent that $\vec{\Omega} \times \vec{v}\,'$ will always be directed toward the center of curvature of the path (i.e., 90° clockwise from $\vec{t}$), and if this direction be denoted by the unit normal vector $-\vec{n}$ ($\vec{n}$ positive in the direction 90° counterclockwise from $\vec{t}$), the force equation (62) may be written

$$\frac{\vec{f}}{m} = \vec{a}\,' - \Omega^2\vec{r} - 2\Omega\, v'\vec{n}\,. \tag{63}$$

If $\vec{t}$ is a unit tangent vector in the direction of $\vec{v}\,'$ and θ^* is the inclination of the trajectory with respect to the original horizontal, h^*h^*, the apparent acceleration $\vec{a}\,'$ may be expressed in terms of the components tangential and normal to the path in the following manner.

By definition

$$\vec{a}\,' = \frac{d}{dt}\left(\vec{t}v'\right) = \vec{t}\,\frac{dv'}{dt} + v'\,\frac{d\vec{t}}{dt}\;. \tag{64}$$

The quantity $d\vec{t}/dt$ may be expressed in the form

$$\frac{d\vec{t}}{dt} = \frac{d\vec{t}}{d\theta^*}\,\frac{d\theta^*}{dt}\;.$$

Since $|d\vec{t}| = |d\theta^*|$, and since $d\vec{t}$ is normal to the trajectory, it is convenient to define $d\vec{t}$ as positive when it corresponds to a positive value of the differential $d\theta^*$. Hence $d\vec{t}/d\theta^*$ will be a unit normal vector which, for the trajectory treated here, will be identical with $\vec{n}$. If θ' is the inclination of the trajectory with respect to the instantaneous horizontal $h'h'$, Fig. 6, the instantaneous angular speed of rotation ω' of the instantaneous horizontal is $\omega' = v'\cos\theta'/r$, and it follows that

$$\frac{d\theta^*}{dt} = \frac{d\theta'}{dt} - \frac{v'\cos\theta'}{r}\;. \tag{65}$$

Hence the expression for $\vec{a}\,'$ becomes

$$\vec{a}\,' = \frac{dv'}{dt}\,\vec{t} + \left(v'\,\frac{d\theta'}{dt} - \frac{v'^2\cos\theta'}{r}\right)\vec{n}\,. \tag{66}$$

Noting that in the total force equation (63) the vector $-\Omega^2\vec{r}$ may be resolved into the components $-r\,\Omega^2 \sin\theta'$ in the direction of $\vec{t}$ and $-r\Omega^2\cos\theta'$ in the direction of $\vec{n}$, Eq. (63) may then be expressed in the following form, entirely in terms of the unit vectors $\vec{t}$ and $\vec{n}$.

$$\frac{\vec{f}}{m} = \left(\frac{dv'}{dt} - r\Omega^2 \sin\theta'\right)\vec{t}$$

$$+ \left(v'\frac{d\theta'}{dt} - \frac{v'^2\cos\theta'}{r} - r\Omega^2\cos\theta' - 2\Omega v'\right)\vec{n} \quad . \tag{67}$$

Hence the true force f_t in the tangential direction $\vec{t}$ and the true force f_n in the direction of the positive normal to the trajectory $\vec{n}$ are

$$\frac{f_t}{m} = \frac{dv'}{dt} - r\Omega^2 \sin\theta', \text{ and} \tag{68}$$

$$\frac{f_n}{m} = v'\frac{d\theta'}{dt} - \frac{v'^2\cos\theta'}{r} - r\Omega^2\cos\theta' - 2\Omega v' \, . \tag{69}$$

2. Application of the Equations of Motion to the Satellite Rocket

The following quantities are now introduced.

F = total thrust of all rocket motors when all of the thrust is in the axial direction

F_c = total magnitude of the thrust of the rocket control motors

D = drag

W = weight based on standard sea-level gravity = mg_s

g_s = standard sea-level value of apparent gravity = 32.174 ft/sec^2

g = absolute gravity

L_a = aerodynamic lift, positive in the direction of $+\vec{n}$

α = angle of tilt of vehicle relative to direction of trajectory, positive in the direction of $+\vec{n}$

f_{tc} = difference between $F_c\cos\alpha$ and the component of the thrust of the rocket control motors in the direction $\vec{t}$

f_{nc} = component of rocket control motor thrust normal to the trajectory, positive in the direction of $+\vec{n}$

Resolving the forces acting on the rocket into components in the direction of $\vec{t}$ and $\vec{n}$, we have

$$f_t = F\cos\alpha - D - mg\sin\theta' + f_{tc} \tag{70}$$

$$f_n = +F\sin\alpha + L_a - mg\cos\theta' + f_{nc}\ , \tag{71}$$

and the force equations may then be written

$$\frac{dv'}{dt} = -\left(g - r\,\Omega^2\right)\sin\theta' + \frac{F}{m}\cos\alpha - \frac{D}{W}g_s + \frac{f_{tc}}{m}\ ,\ \text{and} \tag{72}$$

$$v'\frac{d\theta'}{dt} = -\left(g - r\,\Omega^2\right)\cos\theta' + \frac{v'^2}{r}\cos\theta' + 2\Omega v' + \frac{F}{m}\sin\alpha + \frac{L_a}{W}g_s + \frac{f_{nc}}{m}. \tag{73}$$

It will be noted that weight is defined on the basis of the constant standard sea-level value for the acceleration of gravity, g_s, and therefore does not vary with height. The absolute acceleration of gravity g varies with the distance r from the center of the earth according to the formula

$$g = g_R\left(\frac{R}{r}\right)^2\ , \tag{74}$$

where R is the radius of the earth and g_R is the absolute value of gravity at sea level. The absolute value g_R is the measured or apparent gravity at sea level plus the centripetal acceleration due to the earth's rotation. At the equator g_R = 32.199 ft/sec², the apparent gravity being 32.088 ft/sec².

In these equations F is the combined thrust of all the motors, including the control motors when they are aligned in the axial direction. If the control motor deflections are small, which is a control motor design criterion, then f_{tc} is negligible in Eq. (72) compared to $F\cos\alpha$ and it will therefore be omitted in the following development. On the other hand L_{cf} is not negligible compared to $F\sin\alpha$ in Eq. (73) and consequently may not be disregarded. The value of L_a and the required value for f_{nc} depend to a great extent on the aerodynamic characteristics of the rocket[6], and both f_{nc} and L_a are usually smaller than $F\sin\alpha$. If it now be assumed that the servo-control system is capable of supplying, through a program on the independent parameter α, any reasonable desired program on the net value of $F\sin\alpha + L_a + f_{nc}$, it becomes convenient to define a new independent parameter α^* by the relation

$$F\sin\alpha^* = F\sin\alpha + L_a + f_{nc}\ , \tag{75}$$

so that the determined lift program shall henceforth be specified through a program on the effective angle of attack α^*. It will be noted that the condition $\alpha = \alpha^*$ corresponds to the case of no aerodynamic or control forces. When a program on α^* is determined from considerations in this section, it can be converted to a program on α by means of Eq. (75) as discussed in detail in the aerodynamics report, Ref. 6. When aerodynamic forces are absent and α^* is constant, α^* is practically equal to α, differing from it only because f_{nc} must still supply the moment necessary to turn the rocket so that the condition of constant α^* is maintained along the trajectory. In order to simplify some of the calculations to follow, we shall seek optimum α^* programs by representing the variation of α^* with time by means of a step function. Such a program could not of course be supplied by the servo-control system so that it must be remembered that the program finally chosen must be suitably 'rounded off'. There still exists a cos α term in Eq. (72) which is practically unity for values of α in the range $0 - 3°$ as in this study. Thus, although the approximation $\cos \alpha \doteq 1$ might be used in this equation, it is somewhat more accurate to employ the approximation $\cos \alpha = \cos \alpha^*$, which will be done here. By neglecting f_{tc} and letting $\cos \alpha = \cos \alpha^*$ in Eq. (72) the error in the final velocity is less than 10 ft/sec. It is not permissible to make any simplifying approximations in Eq. (73) and in this equation α^* must be used in accordance with the exact definition, Eq. (75). By making the above simplifications in Eq. (72) and using α^* accurately in Eq. (73), the present trajectory analysis has been divorced from some of the less known aerodynamic characteristics and all of the rigid body dynamics and servo-control interaction. The modified differential equations truly describe the motion of a point mass.

Certain of the forces appearing in the equations of motion may now be replaced in terms of the performance parameters discussed in Part I. We first return to Eq. (7) and, assuming operation with constant rate of mass flow of propellants (dm/dt = const.), evaluate the quantity F/m for any time t within the burning period t_b. It follows from (7) that

$$\frac{F}{m} = -g_s I \frac{1}{m}\frac{dm}{dt} \quad . \tag{76}$$

Also, since dm/dt is constant, it follows that $-dm/dt = W_p/g_s t_b$ and therefore that $m = W_i/g_s - (W_p/g_s t_b)t$, where W_i is the gross weight at the beginning of the burning period and W_p is the total weight of propellants consumed during the burning period t_b. Thus

$$\frac{F}{m} = \frac{g_s I \nu}{t_b\left(1 - \nu \dfrac{t}{t_b}\right)} \, , \tag{77}$$

$$\text{where} \quad \nu = \frac{W_p}{W_i} \, .$$

A refinement of the analysis may be made by considering the weight of fuel used in furnishing power for various auxiliary purposes such as propellant pumps and servomechanisms. The weight W_p is used to denote only the weight of propellants consumed in obtaining thrust from the rocket motor. Let W_{p} denote the weight of fuel used during a burning period for auxiliary purposes and introduce the factor ϵ defined by

$$W_p^* = W_p + W_{\mathrm{p}} = \epsilon W_p, \quad \text{or} \quad \epsilon = 1 + \frac{W_{\mathrm{p}}}{W_p} \, , \tag{78}$$

where $W_p^* = \epsilon W_p$ is equal to the total weight of propellants and fuel consumed in the burning period t_b. This is equivalent to defining a total propellant-gross weight parameter ν^* such that $\nu^* = \epsilon\nu$, see Eq. (19), and when the extra weight of auxiliary propellants is taken into account, Eq. (76) must be written

$$\frac{F}{m} = \frac{g_s I \nu}{t_b\left(1 - \epsilon\nu \frac{t}{t_b}\right)}. \tag{79}$$

In general, if W_i denotes the initial gross weight for a given stage and W denotes the gross weight at any time t during the burning period t_b of the stage, we have

$$\frac{W}{W_i} = \frac{W_i - W_p{}^* \frac{t}{t_b}}{W_i} = 1 - \frac{W_p{}^*}{W_i}\frac{t}{t_b} = 1 - \epsilon\frac{W_p}{W_i}\frac{t}{t_b} = 1 - \epsilon\nu\frac{t}{t_b}. \tag{80}$$

Using this relation, the drag and lift terms in the equations of motion are evaluated by means of the relations

$$\frac{D}{W} = \frac{C_D A q}{W_i\left(1 - \epsilon\nu\frac{t}{t_b}\right)}, \text{ and} \tag{81}$$

$$\frac{L_a}{W} = \frac{C_L A q}{W_i\left(1 - \epsilon\nu\frac{t}{t_b}\right)}, \tag{82}$$

where C_D and C_L are the drag and lift coefficients, A is a representative area (maximum cross section) of the rocket stage, and q is the dynamic pressure. The coefficients C_D and C_L are discussed in Ref. 6. With the aforementioned simplifications, the equations of motion (72) and (73), referred to the rotating system of coordinates, may now be written

$$\frac{dv'}{dt} = \frac{g_s I \nu \cos\alpha^*}{t_b\left(1 - \epsilon\nu\frac{t}{t_b}\right)} - \left[g_R\left(\frac{R}{r}\right)^2 - r\Omega^2\right]\sin\theta' - \left(\frac{D}{W}\right)g_s, \text{ and} \tag{83}$$

$$v'\frac{d\theta'}{dt} = \frac{g_s I \nu \sin\alpha^*}{t_b\left(1 - \epsilon\nu\frac{t}{t_b}\right)} - \left[g_R\left(\frac{R}{r}\right)^2 - r\Omega^2\right]\cos\theta' + \frac{v'^2}{r}\cos\theta' + 2\Omega v', \tag{84}$$

where the gravity relation Eq. (74) has been introduced.

3. Integration of the Equations of Motion

In general these equations of motion cannot be integrated analytically. An exception to this, however, occurs in the highly specialized case of coasting at high altitudes where the lift and drag forces are negligible; in this case, the equation may be integrated to give motion in an elliptical orbit as discussed later. Thus, in order to obtain accurate results, it is necessary to solve the differential equations of motion (83) and (84) by numerical methods. Many such methods are available[7],[8], differing considerably in complexity and amount of labor involved. Since the main term in the equations of motion is the thrust term which is essentially integrable, a rather simple method of successive approximations is used in this study which is based upon a combination of the method of Picard (Ref. 7, pp. 218 to 225) and the Cauchy-Lipschitz process (Ref. 8, Chap. XIII). The equations (83) and (84) apply to any burning period t_{b_j} (i.e. to any stage) and to any time interval Δt within a burning period. Since the method of successive approximations requires that each burning period be broken up into small time intervals, to avoid complicated notation the particular stage in question (the subscript j) will not be specified. However, it is necessary to specify the finite differences within a burning period, and these will be denoted by the subscript k. Considering Eq. (83) for example, the formal integration process would be indicated by

$$\Delta v'_k = \int_{t_1}^{t_2} dv' = g_s \int_{t_1}^{t_2} \frac{I\nu \cos\alpha^*}{t_b\left(1 - \epsilon\nu \frac{t}{t_b}\right)} dt$$

$$- \int_{t_1}^{t_2} \left[g_R\left(\frac{R}{r}\right)^2 - r\,\Omega^2\right] \sin\theta' dt - g_s \int_{t_1}^{t_2} \left(\frac{D}{W}\right) dt \,, \tag{85}$$

where $\Delta v'_k = v'_{k_2} - v'_{k_1}$ is the change in velocity in the kth time interval $\Delta t_k = t_2 - t_1$. If the time interval Δt_k is taken small enough, which is always the case when the numerical method is employed, certain of the quantities vary so slowly with time, and therefore altitude, that it is sufficient to replace them by their mean values during the small time interval Δt, whereby they are then treated as constants. In this way, Eq. (85) is replaced by the expression

$$\Delta v'_k = \frac{g_s \overline{I_k \cos\alpha^*_k}}{\epsilon} \log \frac{1 - \epsilon\nu \frac{t_1}{t_{b_k}}}{1 - \epsilon\nu \frac{t_2}{t_{b_k}}} - \overline{\left[g_R\left(\frac{R}{r}\right)^2 - r\,\Omega^2\right]_k \sin\theta'_k} \cdot \Delta t_k$$

$$- \left(\overline{\frac{D}{W}}\right)_k g_s \cdot \Delta t_k \,, \tag{86}$$

and Eq. (84) becomes

$$\Delta\theta'_k = \frac{g_s}{\epsilon}\overline{\left(\frac{I_k \sin \alpha_k^{*}}{v'_k}\right)} \log \frac{1 - \epsilon\nu \frac{t_1}{t_{b_k}}}{1 - \epsilon\nu \frac{t_2}{t_{b_k}}} - \overline{\left[\frac{g_R \left(\frac{R}{r}\right)^2 - r\Omega^2}{v'}\right] \cos \theta'_k} \cdot \Delta t_k$$

$$+ \overline{\frac{v'_k}{r_k} \cos \theta'_k} \cdot \Delta t_k + 2\Omega \cdot \Delta t \ . \tag{87}$$

A third equation defining the height $h = r - R$ must now be introduced; this is $dh/dt = v \sin \theta'$ which is used in the finite difference form

$$\Delta h_k = \overline{v'_k \sin \theta'_k} \Delta t_k \ , \tag{88}$$

where $\Delta h_k = r_{k_2} - r_{k_1}$. The numerical method of successive approximations is now applied to the equations (86), (87), and (88).

In carrying out the numerical solution, the procedure is to begin with Eq. (86) and calculate $\Delta v'_k$ using a sufficiently small value for Δt_k and using either known initial values or estimated values for the mean value quantities. This gives a value for $\overline{v'_k}$ which is then used in Eq. (87) to get $\Delta\theta'_k$. This will give a value for $\overline{\theta'_k}$ which, together with $\overline{v'_k}$, is used in Eq. (88) to compute the corresponding value for Δh_k. Then using these new values of $\overline{h_k}$ (i.e., $\overline{r_k}$) and $\overline{\theta'_k}$, the process is repeated several times until the values of v', θ', and h converge. In using this method precautions must be taken that the time interval Δt_k is not too large. For even though convergence is secured, if the time interval is not small enough, the final converged values obtained may still differ from the true values by an amount greater than the required accuracy. One simple test of a very practical nature is to compare the converged values based on a given Δt_k with those obtained when Δt_k is taken half as large. If the difference of the two sets of converged values is less than the required accuracy, it may be concluded that the original time interval was not too large. The accuracy which has been specified in most of the calculations is about 10 miles in total height $\Sigma\Delta h_k$ and about 50 ft/sec in total velocity change $\Sigma\Delta v'_k$.

4. The Trajectory Calculations

Various results which were discussed in a general way in Part I will now be treated in more detail. Sufficient discussion has already been given in Part I to show that the best that can be done for a rocket with independent staging is to use a constant value of n throughout the non-coasting part of the trajectory, and also that the use of constant ν corresponds to conditions very near the optimum. Accordingly all calculations are carried out on the basis of n and ν constant as far as staging is concerned. The purpose of the trajectory investigations is to determine, on this basis, the particular trajectory which will satisfy the required orbital conditions with the least value for gross weight, which in most cases means the least value for ν. The basic value for the specific impulse I is determined by the type of propellants used and the combustion temperature as described in Ref. 9, and is to be treated in the trajectory equations as a quantity specified in advance, although a variation will be allowed to take account of the effects of changes with height of the free-air pressure (see Ref. 6) and the changes in design from stage to stage. Likewise, since the general features of the shape of the rocket are known, the value of the drag coefficient C_D, and its variation with speed and altitude, may be considered as known in the trajectory equations (see Ref. 6).

Thus, calculations must be made for various trajectories which differ in the tilt program $\alpha(t)$, in the time τ which specifies where the burning period is interrupted and coasting begins, in the amount of coasting Δh_c, and in the maximum load factor n which enters the trajectory equations implicitly through the burning period, Eq. (31). Consistent with the required orbital conditions (height, velocity, and angle), the problem is to determine, by means of the trajectory equations, the optimum values for least ν of $\alpha^*(t)$, Δh_c, and τ for various values of n. Then, from these values, an optimum n may be chosen corresponding to a minimum value for gross weight by making use of the weight studies of Ref. 3.

Considering particular orbital conditions which must be satisfied by means of the trajectory calculations, these may be expressed in functional form by

$$\begin{aligned} v'_{\substack{orb\\req}} &= v'_{\substack{orb\\req}} \left[\alpha^*(t),\ \Delta h_c,\ \tau,\ n,\ \nu \right] \\ \theta'_{\substack{orb\\req}} &= 0 = \theta'_{orb} \left[\alpha^*(t),\ \Delta h_c,\ \tau,\ n,\ \nu \right] \qquad (89) \\ h_{\substack{orb\\req}} &= h_{\substack{orb\\req}} \left[\alpha^*(t),\ \Delta h_c,\ \tau,\ n,\ \nu \right] . \end{aligned}$$

The orbital angle condition $\theta'_{orb} = 0$ must always be satisfied on a circular orbit, and the angle of inclination must be zero in both the fixed and rotating coordinate systems. In fact, this is the only value for which the angle will have the same value in both systems of coordinates. The orbital velocity is determined mainly through the effect of ν, the orbital angle through the effect of α, and the orbital height through the effect of Δh_c. Strictly speaking the number of stages N should be treated as still another variable, so that all of the following discussion should, in principle, include consideration of variations in N. However, the best value for N is more readily determined by means of the simplified performance equation, Part I, which indicated that most of the trajectory shape study should be carried out on the basis of $N = 3$, but that some consideration should also be given to the rocket with $N = 2$. In view of this result it is not necessary to consider variations with N in the following discussion, since N will always have a fixed value of either 3 or 2.

The problem of determining the best form of a continuous function $\alpha^*(t)$ is essentially a problem in the calculus of variations. However, this problem can be simplified here since it has been found by actual calculations that it is of no value to have any tilt in the higher portions of the trajectory and that the most important factor associated with tilt is the position, or time, in the lower part of the trajectory at which tilting begins. To specify this, the parameter λ is introduced which is defined as the fraction in time of the first burning period t_b at which α first has a value different from zero. Since it is inefficient to use tilt in the higher portions of the trajectory, the tilt angle α must go to zero again somewhere along the trajectory. To determine where the tilting should end and $\alpha = 0$ again, the effect of ending tilt at various positions in the trajectory was investigated over a wide range extending from the middle of the first burning period to the end of the trajectory. It was found that the performance of the rocket and the resulting value for ν were, within a wide range, practically independent of where the tilting was ended. Actually it was found that when the tilt ended early, slightly larger values of α were required in the first part of the trajectory, but that the net effect on the required value of ν was negligible. Accordingly, all of these results indicate that it is quite satisfactory to employ the following simplified $\alpha^*(t)$ program.

> Start the tilt program at the time λ early in the first stage. Keep the effective angle of tilt α^* constant up to 0.9 of the first burning period ($0.9\ t_{b_1}$) and then keep $\alpha^* = 0$ thereafter.

Near the very beginning of the first stage and in the last stage α^* is practically equal to α due to the small magnitude of the aerodynamic forces. As pointed out before, Part I, section 12, the required lift force will always be negative (in the direction toward the center of curvature of the trajectory) and accordingly the tilt angle α is always negative.

Consider further the required orbital conditions, Eqs. (89). The method of calculation which will be used here does not allow one to arrive at specified orbital conditions directly. Rather, the calculations proceed on a somewhat indirect basis, since it is noted that for any chosen set of values for $\alpha^*(t)$, Δh_c, τ, n, and ν, a definite value will result for the velocity at the end of the trajectory. If this velocity is high enough it will correspond to the velocity of some type of orbit but not necessarily to a circular one. Moreover, the angle of inclination θ' at the end of the trajectory will not necessarily give the proper direction of motion required for establishing a circular orbit. To meet the conditions required for establishing a circular orbit, the velocity v_F at the end of the trajectory, at the height h, where h is whatever value results from the calculations, must have the value given by Eq. (57) and the angle θ'_{orb} must be zero. Functionally, the equations (89) show that by proper choice of two of the variables, say α^* and Δh_c, it is possible to satisfy two of the orbital conditions, for example the velocity and angle; when this is done, the third orbital condition is still dependent upon the remaining variable. Thus, in order to find a trajectory which corresponds to any circular orbit at all, it is necessary to have trajectories for various α^* and Δh_c from which, by interpolation, a trajectory can be picked out which corresponds to a circular orbit. If the trajectory curves are comprehensive enough, it will then be possible to find a trajectory which will satisfy the orbital conditions corresponding to a specified height $h_{orb_{req}}$.

Accordingly, we may imagine the elimination of the variables Δh_c and α^* by means of many trajectory calculations, which would then result in a relation of the form $h_{orb} = h_{orb}\ (\lambda,\ \tau,\ n,\ \nu)$ where h_{orb} is any orbital altitude and not necessarily the required one. If the range of ν covered is great enough, one may then specify the value for $h_{orb_{req}}$ and get

$$\nu = \nu(\lambda, \tau, n), \tag{90}$$

where the orbital conditions $v'_{orb_{req}}$, $\theta'_{orb_{req}}$, and $h_{orb_{req}}$ are now considered satisfied and therefore do not enter the functional relation as variables. Having a relation of this type, it would be possible to investigate the variation of ν with λ, τ, and n. However, it is found much more convenient to proceed by first considering relations of the form $h_{orb} = h_{orb}(\lambda, \tau, n, \nu)$ for a given value of ν. Optimum values of λ and τ, for various n, are then found which give the maximum value for h_{orb}. Thus the maximum values of h_{orb} are found rather than the minimum values of ν. This procedure is justified by the following remarks.

In the simplified performance formula (61), the main effect as far as changes in ν are concerned is determined by the term on the left hand side of the equation. The terms in h on the right hand side are nearly independent of λ, τ, and n. The $\overline{\sin\theta}$ term varies with h but its variation with h does not change much with λ and n; that is, when λ and n are held constant, the value of the derivative $(\partial\nu/\partial h)_{\lambda,\tau,n}$ does not depend to any appreciable extent on the particular constant values which λ and n have. On the other hand, the derivative $(\partial\nu/\partial h)_{\lambda,\tau,n}$ does vary somewhat with τ, and this is adequately taken into account in the analysis, as will be seen later. It is found that $(\partial\nu/\partial h)_{\lambda,\tau,n}$ is nearly constant over height intervals of the order of 200 to 300 miles, and the use of this property allows important simplifications to be made. For example, because of this property it is possible to deduce the value of $\nu = \nu(\lambda, \tau, n)$ for a given $h_{orb_{req}}$ from the value of $h_{orb} = h_{orb}(\lambda, \tau, n)$ at constant ν. Use is also made of a further general property that the optimum λ and τ to give the maximum orbital height is the same as the optimum λ and τ to give the minimum ν. These details will become more apparent in the later discussion.

5. Coasting

Before explaining in detail all of the various steps employed in using the trajectory calculations, it will be necessary to discuss the coasting portion of the trajectory. Since there is no rocket motor thrust present during coasting, and since the coasting always occurs at such great heights (100 miles or more) that the drag and lift are negligible, the equations of motion (83) and (84) are considerably simplified and may be integrated. To do this it is convenient for the integration to refer the equations of motion to a fixed (i.e. non-rotating) system of coordinates with origin at the center of the earth, rather than the system of coordinates rotating with the earth, which has been used up to the present. In the fixed system of coordinates the speed and angle variables will be denoted by v and θ without the primes; h is the same in both systems. In the fixed system of coordinates the only forces acting on the rocket during coasting are the gravity and centrifugal forces, and the equations of motion become

$$\frac{dv}{dt} = -g\sin\theta \tag{91}$$

along the trajectory, and

$$v \frac{d\theta}{dt} = -g \cos\theta + \frac{v^2}{r} \cos\theta \tag{92}$$

normal to the trajectory. As shown in Appendix I, these equations are quite easily integrated yielding the solution

$$v^2 - 2\,gr = v_i^2 - 2\,g_i r_i = 2\,E \quad \text{(constant total energy)} \tag{93}$$

$$\text{and} \quad rv \cos\theta = r_i v_i \cos\theta_i \quad \text{(constant angular momentum)} \tag{94}$$

where the subscript i is used to denote the beginning of coasting, and where E is the total energy per unit mass (kinetic plus potential). These integrals could have been written immediately from the well known dynamical law that the motion of a particle in a conservative field of force, the field of gravity in this case, must always be such that the total energy and the angular momentum are conserved. When the field of force is one of gravity, which varies inversely as the square of the distance, we have the well known case of Keplerian or planetary motion for which the path of the particle is known to be elliptic, parabolic, or hyperbolic, depending on whether the total energy is negative, zero, or positive[10].

When the particle is at rest at infinity, its total energy is zero; and it can be shown[11] that the coasting portion of the trajectory will be elliptic, parabolic, or hyperbolic, depending on whether the actual velocity is less than, equal to, or greater than the velocity it would acquire in falling from rest at infinity under the action of gravity, that is, the escape velocity. Since the circular orbit velocity is obviously less than the escape velocity, the velocity during coasting will likewise be less than the escape velocity, the total energy $2E$ will be negative, and, as shown in Appendix I, the coasting trajectory will be an ellipse such that one focus is situated at the center of the earth. Letting $2E = -U^2$ where U is a positive number having the dimensions of velocity, it is shown in Appendix I that the semi-major axis a of the ellipse is defined by $a = r_i^2 g_i / U^2$ and that the semi-minor axis b is defined by $b = v_i r_i \cos\theta_i / U$. Hence the coasting motion is such that, with the center of the earth located at one focus, the coasting begins at a point i near this focus and ends at a point f in the direction toward the other focus. Denoting the end of coasting by the subscript f, the time of transit between the two points i and f is computed from $\Delta t_c = t_f - t_i$, where the instantaneous time t is related to θ by the formula

$$t = -\frac{a \sin^{-1}\left(\frac{b}{c}\tan\theta\right) + b\tan\theta}{U}, \tag{95}$$

which is derived in Appendix I, Eq. (170). In this expression a is the semi-major axis, b is the semi-minor axis, and c is defined by $c = \sqrt{a^2 - b^2}$.

The changes which take place during coasting may be combined with the values at the beginning of coasting, and the changes after coasting to give the final values attained at the end of the trajectory. Considering the final burst of powered flight after coasting, let the changes which occur over this end portion of the trajectory be denoted by Δv_e, $\Delta\theta_e$, and Δh_e. The changes which occur during coasting will be denoted by Δv_c, $\Delta\theta_c$, and Δh_c. Thus, the final conditions attained at the end of the trajectory, denoted by the subscript F, are

$$v_i + \Delta v_c + \Delta v_e = v_F \tag{96}$$

$$\theta_i + \Delta\theta_c + \Delta\theta_e = \theta_F \tag{97}$$

$$h_i + \Delta h_c + \Delta h_e = h_F \;. \tag{98}$$

In these relations it is assumed that all values have been referred to the fixed system of coordinates. Values computed on the basis of the rotating coordinate system may be referred to the fixed coordinate system by means of the relations

$$\begin{aligned} v \sin\theta &= v' \sin\theta' \;, \\ v \cos\theta &= v' \cos\theta' + r\Omega \;, \\ r &= r' . \end{aligned} \tag{99}$$

In the special case that the final conditions at the end of the trajectory are conditions corresponding to a circular orbit, we have $v_F = v_{orb}$, $\theta_F = \theta_{orb} = 0$, and $h_F = h_{orb} = r_{orb} - R$.

Combining Eqs. (93) and (94) and other elliptical characteristics for the coasting motion and imposing the orbital condition that $\theta_F = 0$ yields[9]

$$\frac{\Delta h_c}{r_i} = \left(\frac{a}{r_i} - 1\right) + \sqrt{\left(1 - \frac{a}{r_i}\right)^2 + \left(1 - \frac{\cos^2\theta_i}{\cos^2\Delta\theta_e}\right)\frac{v_i^{\,2}}{U^2}} \;, \tag{100}$$

where from (97), $\cos\Delta\theta_e = \cos\theta_f$. Using Eq. (93) together with Eqs. (96) and (98) with the imposed orbital conditions $v_F = v_{orb}$, $h_F = h_{orb}$, results in the expression [9]

$$\sqrt{1 + \frac{\Delta h_c}{r}} = \frac{\sqrt{\dfrac{a}{r_i}}\left[\dfrac{\Delta v_e}{U}\sqrt{1 - \dfrac{\Delta h_e}{r_{orb}}} + \sqrt{2\left(\dfrac{\Delta v_e}{U}\right)^2 - \left(1 + \dfrac{\Delta h_e}{r_{orb}}\right)}\right]}{1 + \left(\dfrac{\Delta v_e}{U}\right)^2} \;. \tag{101}$$

These are two independent equations in Δh_c involving the three orbital conditions v_{orb}, $\theta_{orb} = 0$, and h_{orb}. Having these equations it is now possible to explain in more detail the method employed in the trajectory calculations.

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[9] In Eq. (100) the positive value of the radical must be used, since the negative value would correspond to negative values of Δh_c. The positive value of the radical must also be used in Eq. (101), since otherwise the curves for Eqs. (100) and (101) cannot intersect to give a circular orbital solution.

6. Procedure Followed in the Trajectory Calculations

1. Considering first the final burst of powered flight after coasting compute, for a certain chosen set of values of ν, n, and τ, the changes Δv_e, $\Delta\theta_e$, and Δh_e associated with this end portion of the trajectory. For the orbital heights of interest here (about 350 miles), these values are essentially independent of orbital height. The calculations 1 may be expressed in functional form by the relations

$$\begin{aligned} \Delta v_e &= \Delta v_e(\nu,\ \tau,\ n) \\ \Delta\theta_e &= \Delta\theta_e(\nu,\ \tau,\ n) \\ \Delta h_e &= \Delta h_e(\nu,\ \tau,\ n) \end{aligned} \tag{102}$$

2. For a certain chosen set of values ν, τ, λ, and n, calculate trajectories up to the beginning of coasting using various α^*. The values of ν and τ used are the same as in 1. This is expressed in functional form by

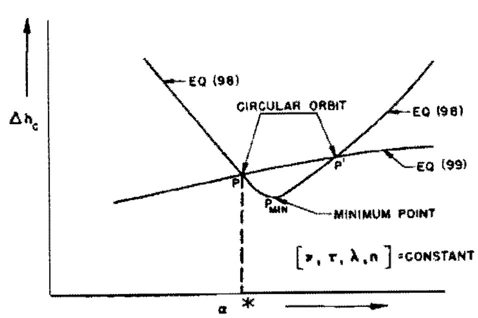
$$\begin{aligned} v_i &= v_i\,([\nu,\ \tau,\ \lambda,\ n],\ \alpha^*) \\ \theta_i &= \theta_i\,([\nu,\ \tau,\ \lambda,\ n],\ \alpha^*) \\ h_i &= h_i\,([\nu,\ \tau,\ \lambda,\ n],\ \alpha^*)\ , \end{aligned} \tag{103}$$

where it is to be understood that the calculations are carried out for certain chosen values of the quantities occurring within the bracket. The quantity outside the bracket may be viewed in the manner of a running variable such that it varies over a range of values for each set of conditions specified within the bracket. Unless it is specifically understood otherwise, as in coasting and in the final burst of powered flight, all values are first computed in the system of coordinates which rotates with the earth and are then transformed to the fixed system of coordinates.

3. Using the results (102) and (103) in the coasting equations (100) and (101) gives relations of the form

$$\Delta h_c = \Delta h_c\ ([\nu,\ \tau,\ \lambda,\ n],\ \alpha^*)\ , \tag{104}$$

and plots are made of Δh_c against α^* as shown in Fig. 7.



SCHEMATIC SKETCH TO ILLUSTRATE HOW AN ORBITAL CONDITION IS OBTAINED

FIG. 7

Curve 1 is obtained by always satisfying the conditions of Eqs. (102) and (103) in Eq. (101), while curve 2 is obtained by always satisfying the conditions of Eqs. (102) and (103) in Eq. (100). The set of values (α^*, Δh_c) corresponding to the points P and P' where the two curves intersect will obviously determine a circular orbit and will therefore be denoted by (α^*_{orb}, $\Delta h_{c_{orb}}$). The circular orbit will therefore correspond to a certain height h_{orb} which may be expressed functionally as

$$\Delta h_i + \Delta h_{c_{orb}} + \Delta h_e = h_{orb} = h_{orb}\left([\nu,\ \tau,\ \lambda,\ n],\ \alpha^*_{orb},\ \Delta h_{c_{orb}}\right) . \qquad (105)$$

In this way for any given set of values $[\nu, \tau, \lambda, n]$, a circular orbit may be established at some known height h_{orb} which is not specified in advance and which is therefore not necessarily the orbital height desired. Moreover, the values (ν, τ, λ, n) will not, in general, be the best values to use from the standpoint of minimum gross weight. The point P' nearly always corresponds to negative θ_i and to ranges which are more than halfway around the earth. Moreover, although Δh_c is higher, h_i is lower for P' than for P causing the rocket to be subjected, usually, to more drag during coasting which tends to make the calculations by which P' is reached erroneous to the extent that h_{orb} for P' may well be lower than that for P.

4. Steps 1, 2, and 3 are now repeated holding $[\nu, \lambda, n]$ constant but using various τ. This process may be indicated by the notation

$$h_{orb} = h_{orb}\left([\nu,\ (\tau),\ \lambda,\ n],\ \alpha^*_{orb},\ \Delta h_{c_{orb}}\right) , \qquad (106)$$

where the symbol (τ) is used to indicate that τ varies over a range of values while the other numbers ν, λ, n remain constant. From the process indicated by (106) a certain largest physically possible value of τ is found, which may be indicated by τ_{max} and which corresponds practically to the largest value attainable for h_{orb}. That is, the value $\tau = \tau_{max}$ is very close to the value $\tau = \tau_{opt}$ which makes h_{orb} a maximum. In a diagram such as that of Fig. (7), $\tau = \tau_{max}$ corresponds to the case in which curve 1 intersects curve 2 only once, which occurs at the minimum point P_m. The value $\tau = \tau_{opt}$ is such that curve 1 intersects curve 2 twice very close to the minimum. It is not clear in the present study, due to the neglected drag effect, whether the optimum is at P or at P', but it is apparent that P or P' is so near to the minimum that the h_{orb} value for $\tau = \tau_{max}$ and $\tau = \tau_{opt}$ are so nearly the same that we shall assume them approximately equal in this discussion. This problem should be more thoroughly investigated in any study where the emphasis is not on short ranges such as are desired here. This value $\tau = \tau_{opt} \doteq \tau_{max}$ is important since, as discussed in section 13 of Part I, this value of τ leads to the least value for ν. It was pointed out (section 4, Part II) that the value $\tau = \tau_{opt}$ for maximum h_{orb} would give the least value for ν if h_{orb} were fixed or specified in advance. In fact, since it will be found desirable later to have the dependence of $(\partial\nu/\partial h)_{\lambda,\tau,n}$ on τ, it is worthwhile at this stage to repeat the calculations for various ν and in this way obtain the relation between ν and h_{orb} for various τ. This relation immediately gives $(\partial\nu/\partial h)_{\lambda,\tau,n}$ as a function of τ, and also gives τ_{opt} for minimum ν at constant h_{orb}. This is the same τ_{opt} as already discussed for, since

$$\left(\frac{\partial\nu}{\partial\lambda}\right)_{\tau,n,h} = -\left(\frac{\partial h}{\partial\lambda}\right)_{\tau,n,\nu}\left(\frac{\partial\nu}{\partial h}\right)_{\lambda,\tau,n} , \qquad (107)$$

it follows that for the same value of τ both $(\partial\nu/\partial\lambda)_{\tau,n,h}$ and $(\partial h/\partial\lambda)_{\tau,n,\nu}$ are zero. Using the τ_{opt} determined in either of the ways mentioned above and returning to the discussion of the variation of h_{orb} with fixed ν, we may write

$$h_{orb} = h_{orb}\left([\nu,\ \tau_{opt},\ \lambda,\ n],\ \alpha^*_{orb},\ \Delta h_{c_{orb}}\right) . \tag{108}$$

5. Steps 2, 3, and 4 are repeated for various λ giving

$$h_{orb} = h_{orb}\left([\nu,\ \tau_{opt},\ (\lambda),\ n],\ \alpha^*_{orb},\ \Delta h_{c_{orb}}\right) . \tag{109}$$

6. Steps 2, 3, 4, and 5 are repeated for various n giving

$$h_{orb} = h_{orb}\left([\nu,\ \tau_{opt},\ (\lambda),\ (n)],\ \alpha^*_{orb},\ \Delta h_{c_{orb}}\right) . \tag{110}$$

Thus far, Eqs. (108) to (110), the calculations described are still based on the original value chosen for ν which will now be denoted by ν_0. On this basis, the calculations carried up to the stage indicated by (110) give, over a range of values for λ and n, the values of τ_{opt} and the corresponding maximum orbital heights for a circular orbit. Thus, for each of the values (λ) and (n) the calculations determine the value of τ_{opt} and the corresponding maximum value of h_{orb}. It now remains to determine the minimum ν corresponding to the desired orbital height of 350 miles.

7. Corresponding to each of the different values of n, the optimum value of λ ($\lambda = \lambda_{opt}$) is chosen which gives the greatest of the corresponding maximum orbital heights; this may be denoted by $h_{orb_{max}}$. This process may be looked upon as finding a double maximum or the ultimate maximum since for each (n) there is found for a series of values λ the corresponding values of τ_{opt}, and from these λ there is chosen a λ_{opt}. In this way the ultimate maximum orbital height can be determined as a function of n. This relation is then used to find the minimum value of ν, $\nu = \nu_{min}$ (as a function of n), which will just suffice to give the desired orbital height of 350 miles. To do this, use is made of the approximate result that $(\partial\nu/\partial h)_{\tau,\lambda,n}$ is independent of the values λ and n.

Since $\partial\nu/\partial h$ is fairly constant with h we may write

$$\nu_{req} = \nu_0 - \left(\frac{\partial\nu}{\partial h}\right)_{\tau,\lambda,n,}\left(h_{orb} - h_{orb_{req}}\right) , \tag{111}$$

which may be used to calculate the value of ν_{req} for a given $h_{orb_{req}}$ when any h_{orb} is known for a particular ν_0. Using this procedure it is necessary to know $(\partial\nu/\partial h)_{\tau,\lambda,n}$ as a function of τ. It will be seen later, Figure 8, that it is more convenient to use the range instead of the parameter τ.

In particular, for the values $\tau = \tau_{opt}$, Eq. (111) may be used to obtain ν as a function of λ and n from the known relation of h_{orb} to λ and n. Further, for the condition λ_{opt} and τ_{opt}, the relation between ν_{min} and n may be calculated, and from this W_1 may be obtained as a function of n from structural considerations. This last relation will have a minimum which will determine n_{opt}, which in turn will determine a final ν_{min}.

8. The value for ν_{min} as obtained above will be slightly in error owing to the approximate nature of the relationship $(\partial\nu/\partial h)_{\tau,\lambda,n}$ = const. which was used. Therefore a final optimum trajectory must be determined. Using the fixed values τ_{opt}, λ_{opt}, n_{opt}, determined above, trajectories are computed for three or four values of ν having values in the neighborhood of the value ν_{min}. Each of these trajectories will determine an orbital height. By interpolating between these orbital heights for the desired orbital height (350 miles), the corresponding interpolated value of ν is the final minimum value which gives the final optimum trajectory (see for example Fig. 8 discussed later).

The method which has been used here to determine the optimum trajectory (τ_{opt}, λ_{opt}, n_{opt}, ν_{min}) has been based on the determination of maximum orbital heights and on the use of the relation $(\partial\nu/\partial h)_{\lambda,\tau,n}$ = const. Although this relation is only approximate, its use has been found justified since the optima are broad and because actual calculations of $(\partial\nu/\partial h)_{\lambda,\tau,n}$ at different sets of values [λ and n] have verified the approximate constancy of the derivative. The procedure adopted here for the trajectory calculations, although somewhat indirect, allows the optimum trajectory to be found in much less time than if the calculations were carried out in the more direct manner based on the specification in advance of the desired orbital height.

7. Range

Before the results of the trajectory calculations can be discussed with completeness, it will be necessary to make a few remarks concerning the range. The range corresponding to any portion of the trajectory is defined as the distance intercepted on the earth's surface by the two radius vectors extending from the center of the earth to the end points of the portion of the trajectory. Thus the range is determined by projecting the trajectory onto the surface of the earth by means of the verticals at the two end points of the trajectory. Letting ϕ and r be the polar coordinates (with origin at the center of the earth) of a point moving along the trajectory, and letting s_h be the horizontal projection of the trajectory at the distance r and S_h the corresponding projection at the surface of the earth at $r = R$ (the range), it follows that

$$dS_h = R\,d\phi \,,$$

$$ds_h = r\,d\phi \,, \quad \text{and}$$

$$dS_h = \frac{R}{r}\,ds_h \,.$$

Since $ds_h = v \cos\theta\, dt$, the differential equation for the range is

$$dS_h = \frac{R}{r}\,v \cos\theta\, dt \,, \tag{112}$$

from which the range itself is given by the integration

$$S_h = R\int_{t_1}^{t_2} \frac{v}{r}\cos\theta\, dt \ , \tag{113}$$

where $\frac{v\cos\theta}{r}dt = d\phi$. For the non-coasting portions of the trajectory, the range is obtained by numerical integration of Eq. (113).

Since the coasting part of the trajectory is a portion of an ellipse, the range for this part of the trajectory may be obtained analytically by integration or from the analytic geometry of the ellipse. If ϕ_i and ϕ_f represent the polar angle at the beginning and end of coasting, the range during coasting is

$$S_h = R(\phi_i - \phi_f) \ , \tag{114}$$

and it is therefore required to find the expression for the angle ϕ for this part of the trajectory. This expression is derived in Appendix I, Eq. (165), where it is found that

$$\phi = \tan^{-1}\left[\frac{\mu\tan\theta}{\tan^2\theta + (1-\mu)}\right] \ , \tag{115}$$

where

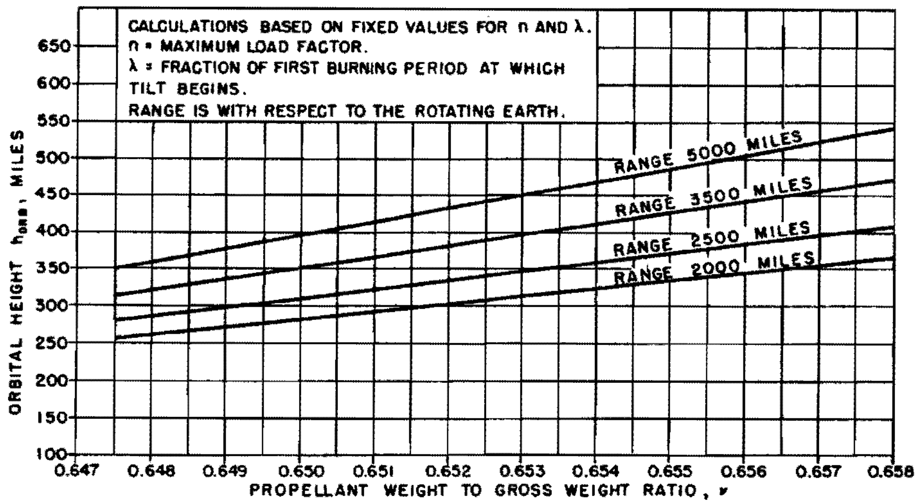
$$\mu = \frac{v^2 r}{g_R R^2} = \frac{v^2}{gr} \ . \tag{116}$$

Thus when θ_i, θ_f, r_i, and r_f are given, the range is computed immediately by use of Eqs. (114) and (115).

It should be pointed out that in this study we are mainly concerned with ranges referred to the rotating earth. This will be denoted by S_h'. The above formulae refer to a non-rotating coordinate system and can be converted to values referred to the rotating earth in the case of the calculations for powered flight by using 'primed' values in Eq. (113), and in the case of coasting by subtracting from Eq. (114) the term $R\Omega(t_f - t_i)$.

Having introduced the range, it may now be noted that to a first approximation the range of the ascending trajectory is determined when τ is given; that is, τ governs range in the main. Since we are more interested in the range (actually we want to limit it to 2500 miles on the earth's moving surface so that communication does not become too difficult) than in τ, it is convenient to replace the independent variable τ by the independent variable S_h'. Since it has been found that $(\partial v/\partial h)_{\tau,\lambda,n}$ varies mainly with τ, it then follows that $(\partial v/\partial h)_{S_h',\lambda,n}$ varies mainly with S_h'. This is

illustrated by Fig. 8 which, for various ranges S_h', shows the variation of ν with h_{orb} for a three stage hydrazine-oxygen satellite rocket.

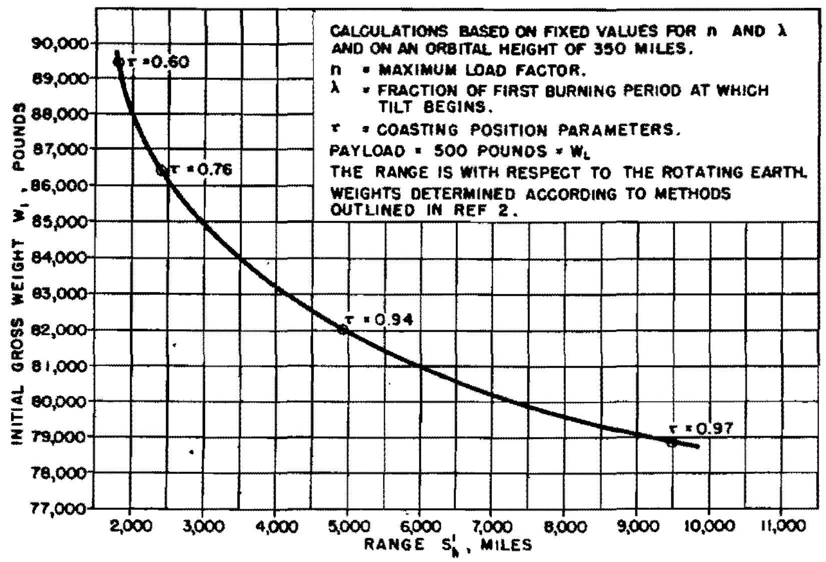


VARIATION OF h_{ORB} WITH ν FOR A THREE STAGE HYDRAZINE — OXYGEN SATELLITE ROCKET.

FIG. 8

8. Results of the Trajectory Calculations for a Three Stage Satellite Rocket

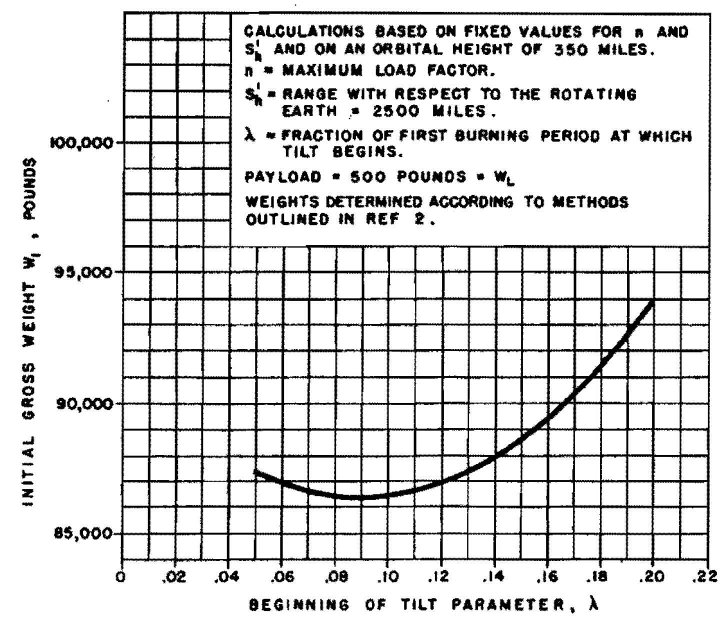
The calculations discussed here refer to a three stage hydrazine-oxygen rocket having the optimum shape as determined by the analysis of Ref. 6. The two stage rocket will be discussed later. Basic to the optima study is the use of $(\partial\nu/\partial h)_{S_h',\lambda,n}$ which must be known for different values of S_h'. Fig. 8 contains a plot of h_{orb} vs. ν for various ranges and therefore determines $(\partial\nu/\partial h)_{S_h',\lambda,n}$ as a function of S_h'. For the three stage rocket the trajectory calculations give $\tau_{opt} = \tau_{max} = 0.97$ for $h_{orb_{req}} = 350$ miles; it is found, however, that this gives a total range extending about halfway around the earth. For greater orbital heights the value of τ_{opt} becomes more nearly unity and in order to escape the earth entirely the value is exactly one. For the purposes of communication it is essential that the total range from launching to establishment of the circular orbit does not exceed about 2500 miles. For this reason it is necessary to choose a smaller (range control) value for τ and it is found that values in the neighborhood of 0.75 give the maximum allowable range of about 2500 miles. This will cause an increase in the gross weight, but the change is small. Using the final values adopted for $(W_B/W)_j$ and W_L together with ν *vs.* S_h' for $h_{orb_{req}}$ from Fig. 8, the variation of gross weight with range may be computed giving the result shown in Fig. 9. Fig. 9 shows the importance of using a large value for τ. It also shows the decrease in weight which would result if ranges of the order of 7000 miles could be allowed.



VARIATION OF GROSS WEIGHT WITH RANGE FOR A THREE STAGE HYDRAZINE — OXYGEN SATELLITE ROCKET.

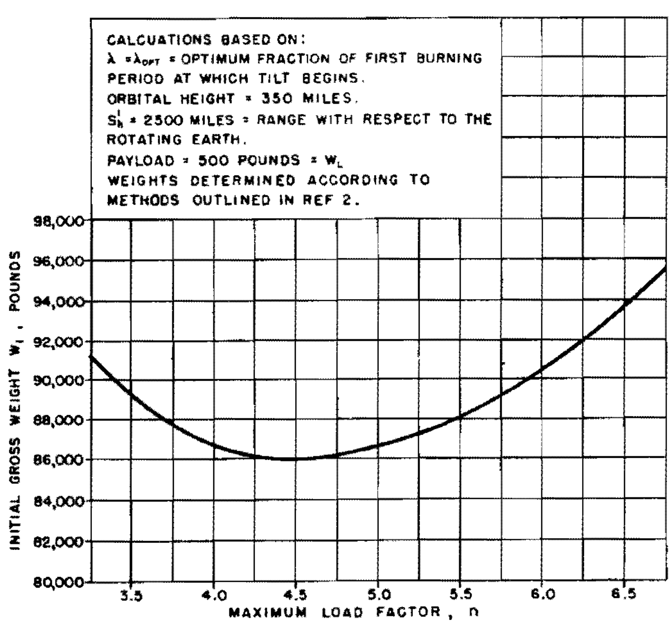
FIG. 9

Having chosen the range and the corresponding approximate value for τ, the variation of h_{orb} with λ and n for this range can be converted by use of $(\partial\nu/\partial h)_{S_h',\lambda,n}$ to a variation of ν with λ and n by using the values of $(\partial\nu/\partial h)_{S_h',\lambda,n}$ derived from Fig. 8. Although these calculations were not as exhaustive as could be desired, they sufficed, nevertheless, to indicate that the variation of ν with λ over the range of n of interest was quite small. Using weights as determined in Ref. 3 for the final rocket design, the variation of ν with λ can be converted to the variation of W_1 with λ giving the results shown in Fig. 10. These results are valid for values of n lying between 4 and 7, and they indicate that a value $\lambda = 0.1$ or somewhat smaller is best. The value $\lambda = 0.1$ was chosen as best because a smaller λ would have caused greater bending of the path and hence greater dynamic pressure q in the latter part of the first stage and in the second stage. This would result in higher temperatures and in more difficult control problems. Using the values of λ_{opt} as determined previously (which vary slightly with n but are approximately equal to 0.1), ν_{min} can be determined as a function of n in a manner similar to that used for λ, and this in turn can be converted through structural considerations into W_1 vs. n. While ν vs. n has a minimum when $n = \infty$, W_1 vs. n has a minimum at about $n = 4.5$ as shown in Fig. 11. However, before the value $n = 4.5$ had been obtained, calculations had already been started using $n = 5.0$ which, at the time, appeared to be the best value. Since, as shown by Fig. 11, the use of $n = 5.0$ instead of 4.5 has a very small effect on the magnitude of W_1, for the hydrazine-oxygen rocket it did not seem worthwhile to start the final calculations over again and they were therefore completed using the value $n = 5.0$.



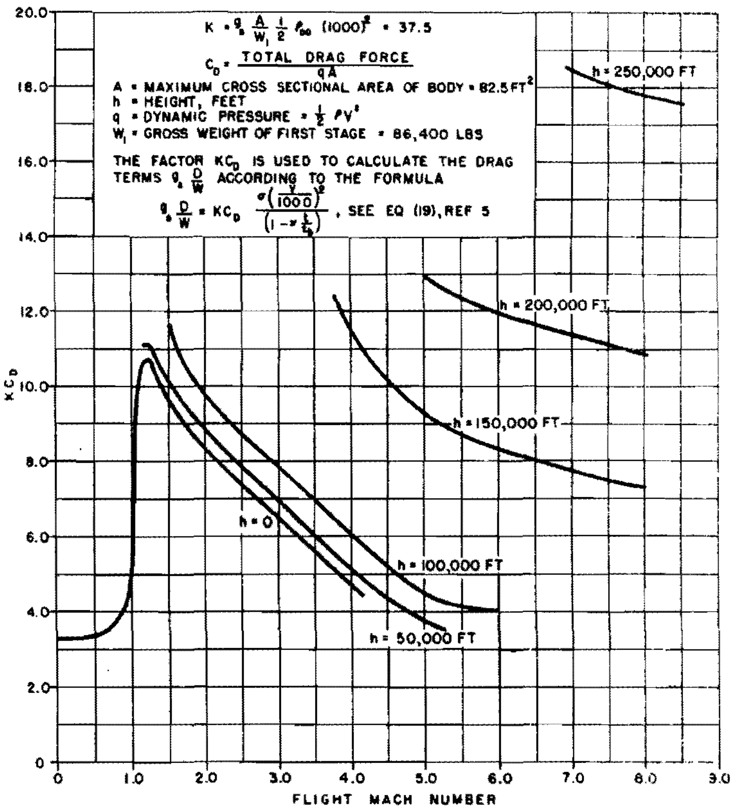
VARIATION OF GROSS WEIGHT WITH THE TIME OF BEGINNING OF TILT FOR A THREE STAGE HYDRAZINE—OXYGEN SATELLITE ROCKET.

FIG. 10



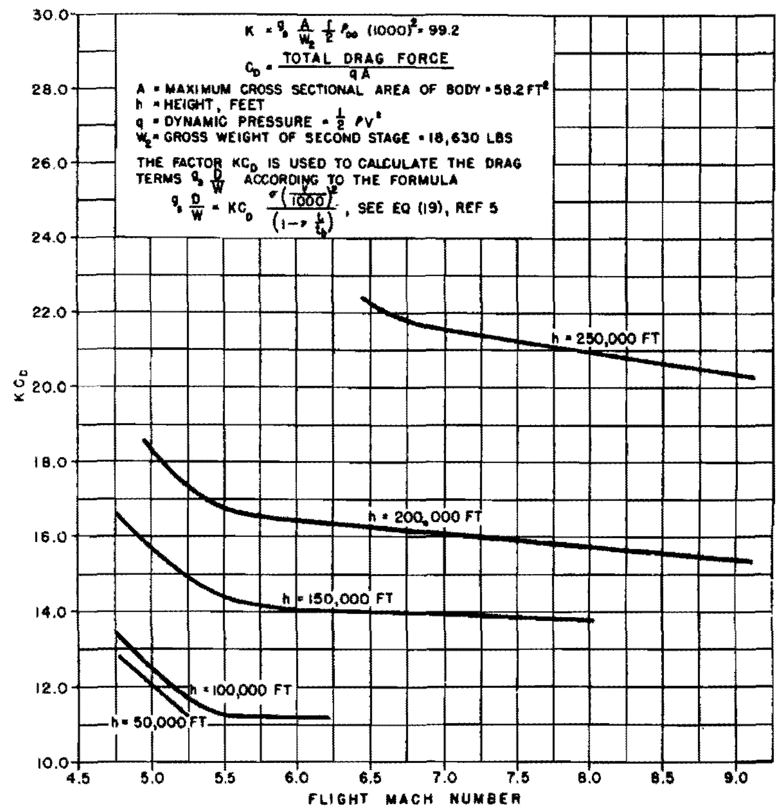
VARIATION OF GROSS WEIGHT WITH THE MAXIMUM LOAD FACTOR FOR A THREE STAGE HYDRAZINE—OXYGEN SATELLITE ROCKET.

FIG. 11



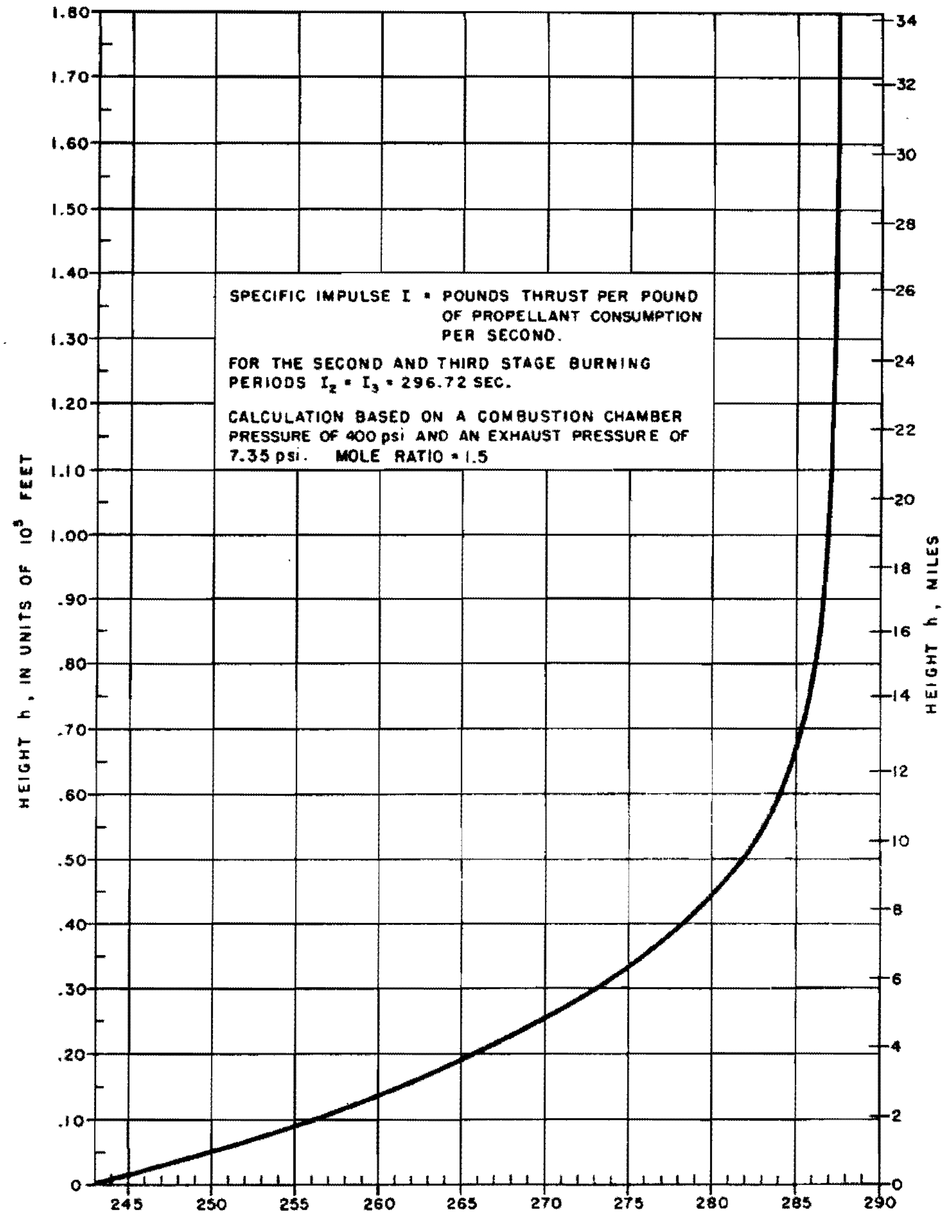
THE DRAG FACTOR KC_D AS A FUNCTION OF HEIGHT AND MACH NUMBER FOR THE FIRST STAGE OF THE THREE STAGE HYDRAZINE-OXYGEN SATELLITE ROCKET.

FIG. 12



THE DRAG FACTOR KC_D AS A FUNCTION OF HEIGHT AND MACH NUMBER FOR THE SECOND STAGE OF THE THREE STAGE HYDRAZINE-OXYGEN SATELLITE ROCKET.

FIG. 12A

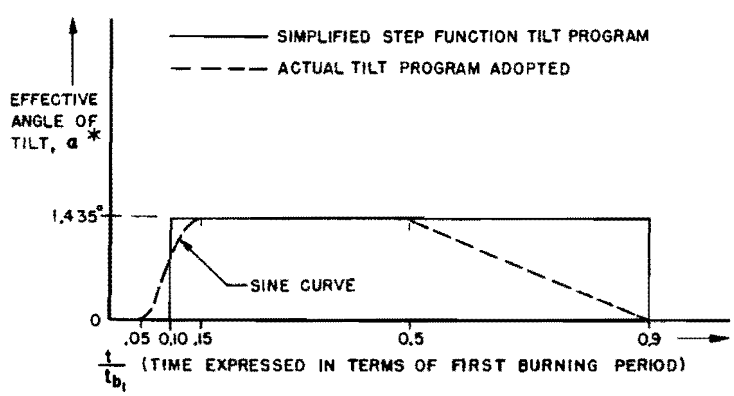


VARIATION OF SPECIFIC IMPULSE WITH HEIGHT DURING THE FIRST STAGE BURNING PERIOD FOR THE HYDRAZINE — OXYGEN PROPELLANT SYSTEM.

FIG. 13

In carrying out the calculations for the final trajectory, use was made of the most recent aerodynamic data[6] (see the drag curve, Fig. 12), atmospheric data[4], and specific impulse data[9]. The final optimum trajectory consistent with the range limitation is based on the values $\lambda = 0.1$, $n = 5$, and $\tau = .762$. The calculations have been based upon specific impulse values corresponding to the hydrazine-oxygen system of propellants, which, from all considerations, appears to be the most practical system to use. The values of specific impulse versus height for the first stage burning period are shown in Fig. 13. In the second and third stages it is sufficient to use the constant value $I_2 = I_3 = 296.7$ seconds.

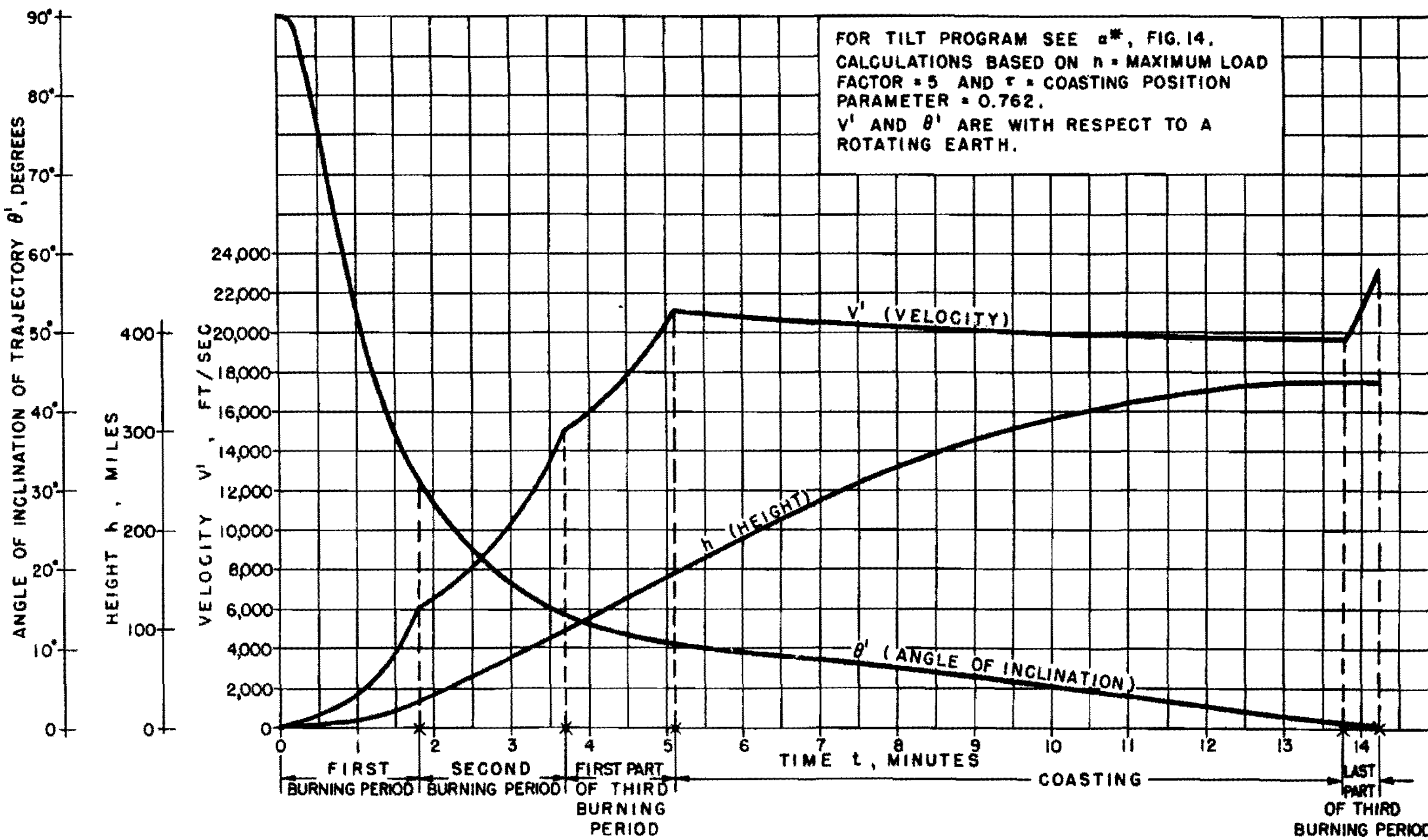
Using the value $\lambda = 0.1$, the simplified tilt program discussed in section 4 would be as shown by the full lines in Fig. 14. Actually, for various control reasons, it was found necessary to adopt a more gradual type of tilt program as indicated by the dotted lines. In this program the tilt was assumed to begin at $\lambda = 0.05$. With time expressed in terms of t/t_{b_1}, the effective tilt α^* was then assumed to vary as a sine curve reaching a maximum value at 0.15, to remain constant at this value up to 0.5, and then to decrease linearly to zero at 0.9. As far as the trajectory is concerned however, the result is practically the same as if the step function tilt program were used with $\lambda = 0.1$. The program of the actual tilt angle α is discussed in Ref. 6.



SCHEMATIC DIAGRAM TO SHOW TILT PROGRAM

FIG. 14

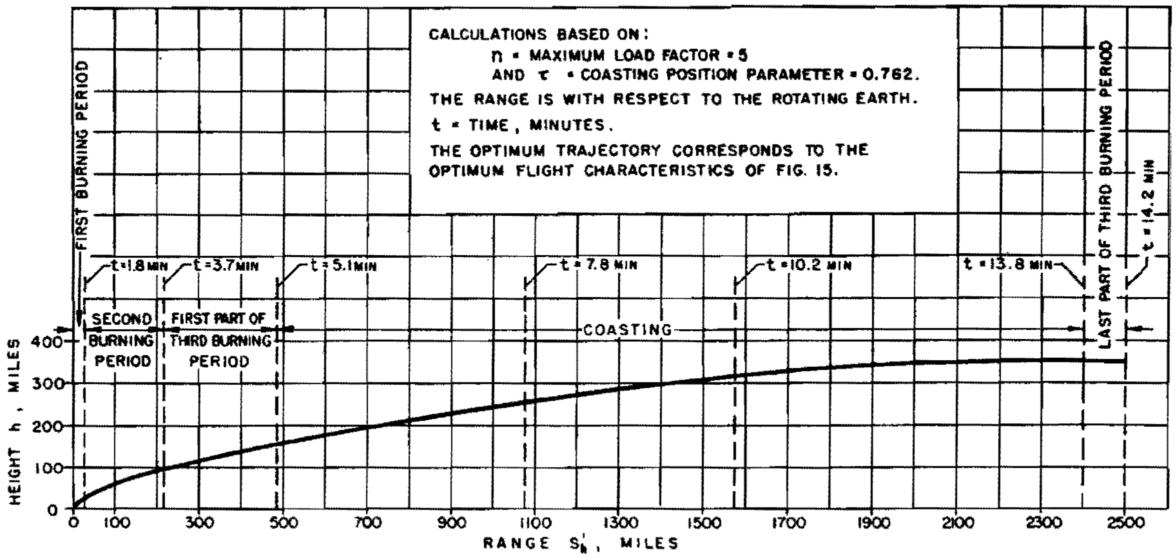
The characteristics of the final optimum trajectory are shown in Figs. 15 and 16. The final value obtained for the minimum required value of the gross weight is $W_1 =$ 84,400 pounds which is based on the final value $\nu_{min} = .6533$ resulting from this study and the final structural considerations. However, since this final minimum value for W_1 became available only at the very end of the study, all curves in this report referring to the three stage hydrazine-oxygen rocket corresponding to the conditions $h_{orb} = 350$ miles and $S_h' = 2500$ miles are based upon the value $W_1 = 86{,}400$ pounds. This value represented the minimum gross weight up to that stage in the analysis where the final optimum trajectory was calculated. The 2000 pound difference in the two values is entirely negligible as far as the trajectory is concerned.



FLIGHT CHARACTERISTICS OF THE OPTIMUM TRAJECTORY FOR A THREE STAGE HYDRAZINE — OXYGEN SATELLITE ROCKET

FIG. 15

February 1, 1947



OPTIMUM TRAJECTORY, HEIGHT VS RANGE, FOR A THREE STAGE HYDRAZINE —OXYGEN SATELLITE ROCKET.

FIG. 16

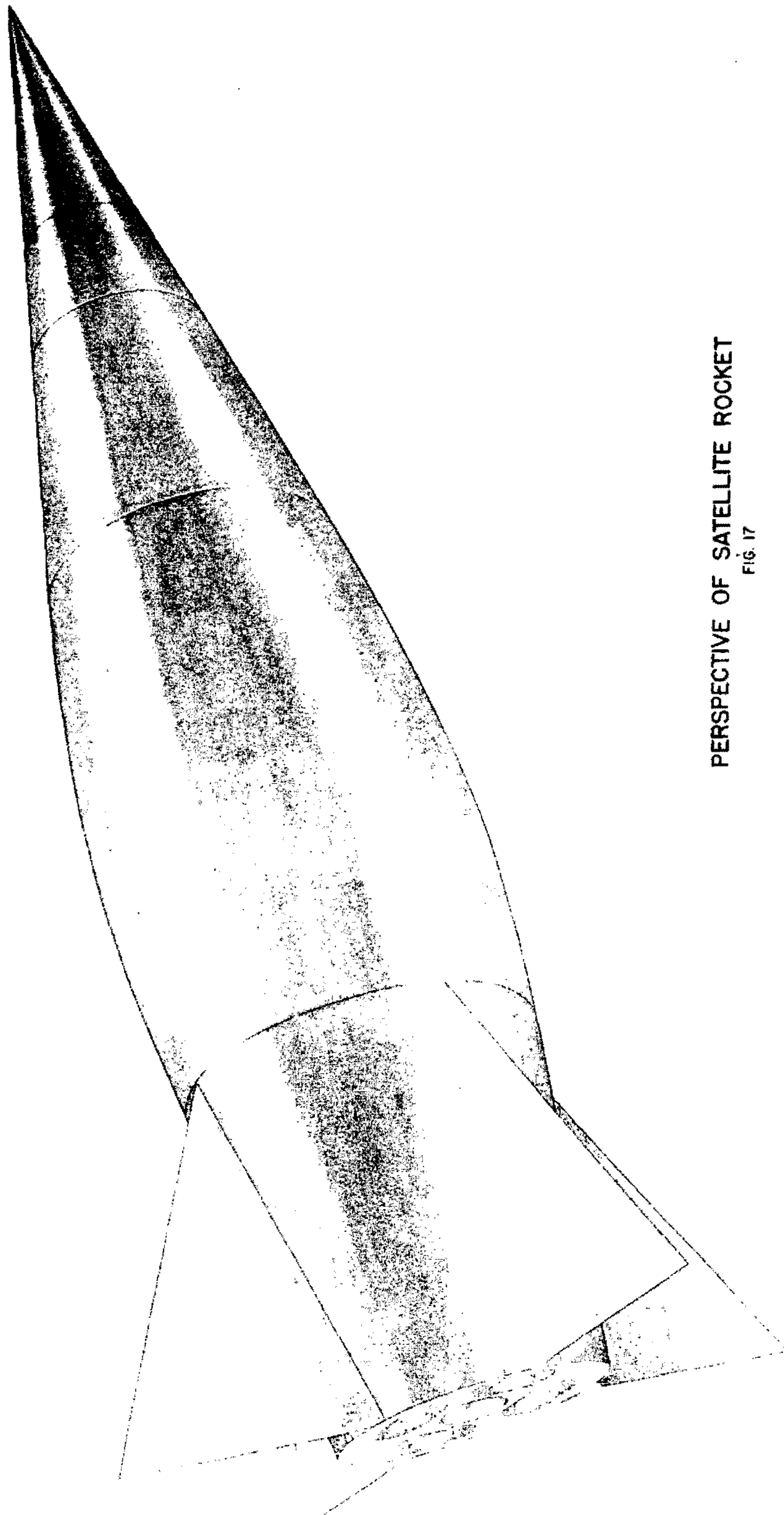
The final optimum design for the rocket corresponding to the optimum trajectory and the value found for the minimum gross weight is shown in Figs. 17 and 17a. The characteristic features of the design are discussed in Refs. 6 and 3.

9. Results of Trajectory Calculations for Rockets Having From Two to Six Stages

Although the calculations for the two stage rocket have not yet been completed, there is no indication from the results available at the present stage of the investigation that the optimum values for τ, λ, n will differ appreciably from those obtained for the three stage rocket. Proceeding on the basis that this result will be borne out by the completed investigation, the minimum value of ν which will give an orbital height of 350 miles may be obtained from the corresponding value $\nu_{3\,min}$ for the three stage rocket by use of the simple formula

$$2 \log \frac{1}{1 - \nu_{2\,min}} = 3 \log \frac{1}{1 - \nu_{3\,min}}, \tag{117}$$

where $\nu_{2\,min}$ refers to the two stage rocket.



PERSPECTIVE OF SATELLITE ROCKET
FIG. 17

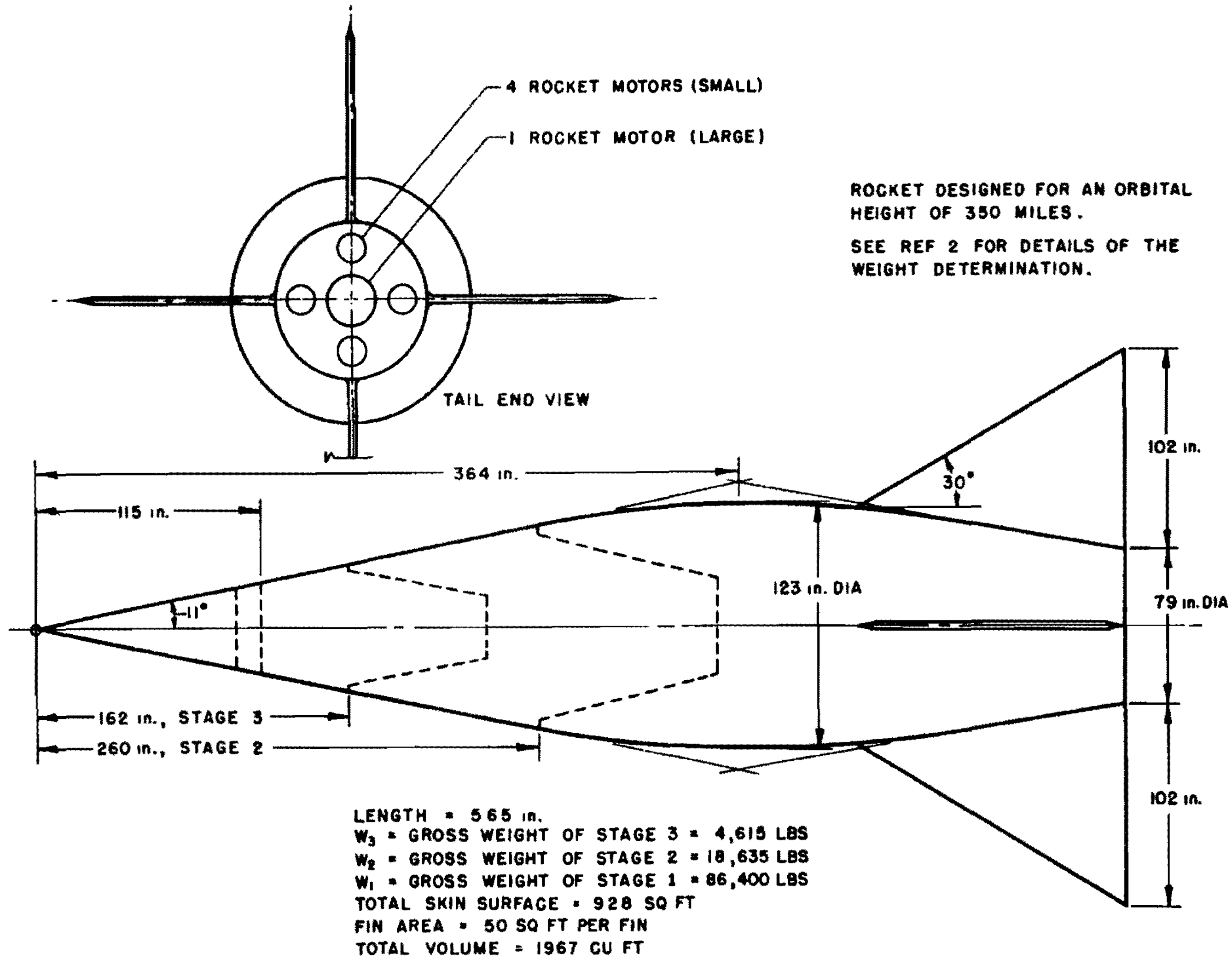


DIAGRAM OF THE THREE STAGE HYDRAZINE-OXYGEN SATELLITE ROCKET.

FIG. 17 A

Very early investigations showed that this simple formula could be used with confidence in deducing comparative values of ν for different rockets varying in number of stages as much as from one to five. The basic reason underlying this result is that optimum trajectory shapes are quite similar regardless of the number of stages. Using the estimates of $(W_B/W)_j$ for a two stage rocket and using the same average specific impulse I (hydrazine-oxygen) as for the three stage rocket, the variation of gross weight with the number of stages is obtained[3] as shown in Fig. 18. These results are based on an orbital height of 350 miles. Curves are also included for the hydrazine-fluorine, alcohol-oxygen, aniline-nitric acid, hydrazine-anhydrous hydrogen peroxide, and hydrogen-oxygen propellant systems. It should be pointed out that the gross weights which have been derived here are based upon a final stage payload (weight of fixed equipment) W_L' of 1080 pounds (satellite payload W_L = 500 pounds). The gross weights derived here may be altered to suit the condition W_L' = 700 pounds simply by multiplying them by the factor 700/1080. Although it is seen from Fig. 18 that the three and four stage hydrazine-oxygen rockets have about the same gross weight, the choice of the three stage rocket is obvious because it is less complex. It is also apparent that the hydrazine-oxygen propellant system, which is the one chosen for the satellite rocket, does not represent the best system which could be used. It is only the practical, but highly important, considerations of handling and availability that have led to the choice of the hydrazine-oxygen system.

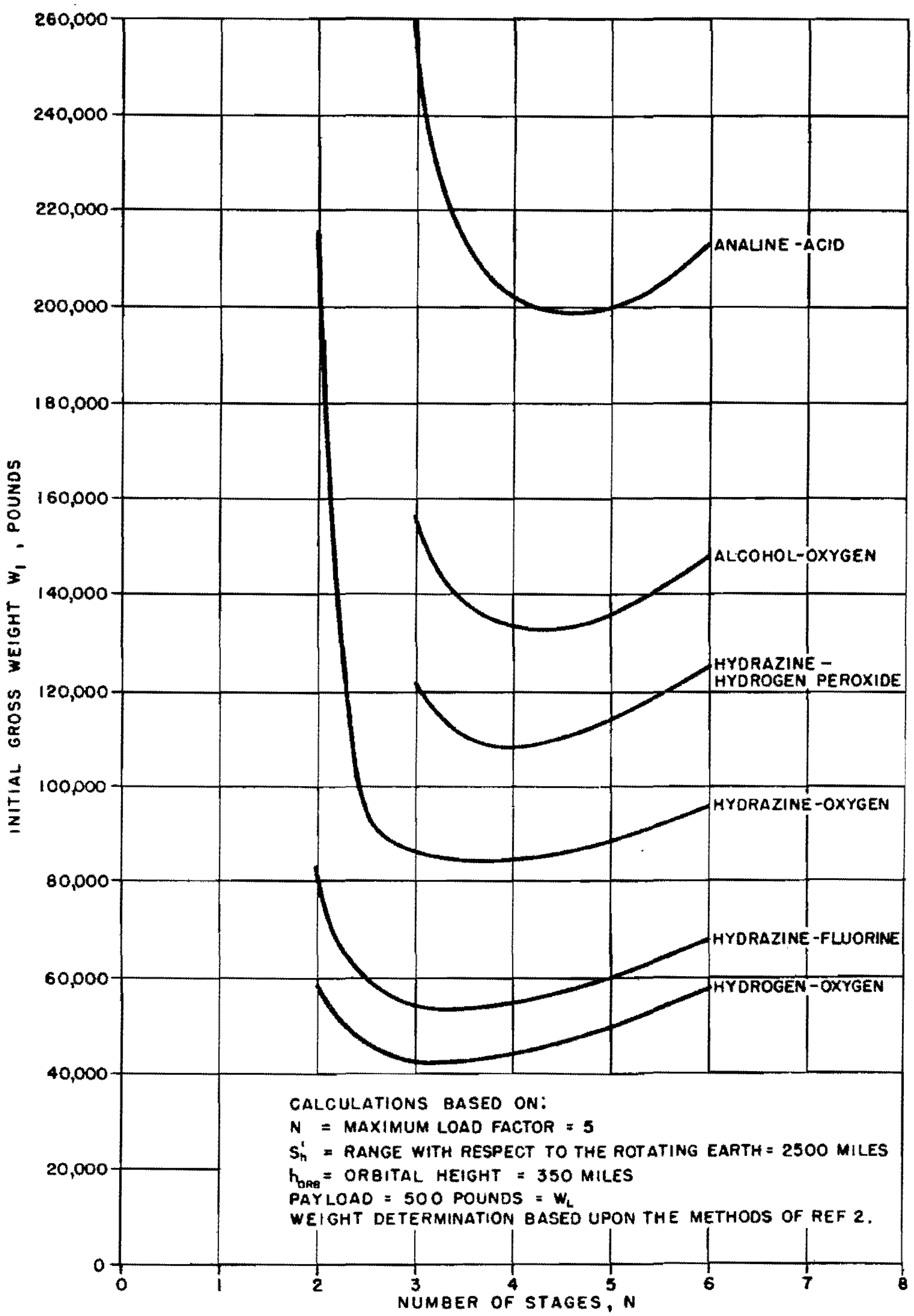
Although the curves of Fig. 18 indicate that a three stage rocket is best when the orbital height is 350 miles, it is found that if higher orbital heights are required the optimum number of stages increases. This is shown by calculating W_1 versus h_{orb} for a three stage hydrazine-oxygen rocket from the results of Fig. 8 and then using the simplified formula

$$3 \log \frac{1}{1 - \nu_{3\,min}} = 4 \log \frac{1}{1 - \nu_{4\,min}}$$

to obtain $\nu_{4\,min}$ from which W_1 versus h_{orb} may be calculated for a four stage rocket. This result is shown in Fig. 19 from which it is seen that at the higher orbital altitudes a four stage rocket requires less gross weight than a three stage rocket. Since the results of Fig. 8 have been greatly extrapolated in order to obtain these high altitude results, the greatest orbital height attainable with a three stage rocket is really higher than that indicated in Fig. 19. Also, since the results were based on a range of 2500 miles, the heights indicated are lower than those which would be obtained for optimum ranges. In connection with these attained orbital altitudes it is interesting to note that the rocket of final design (three stage hydrazine-oxygen rocket of 86,400 pounds gross weight) would be able to attain a maximum height of about 2300 miles by travelling on a perfectly vertical path.

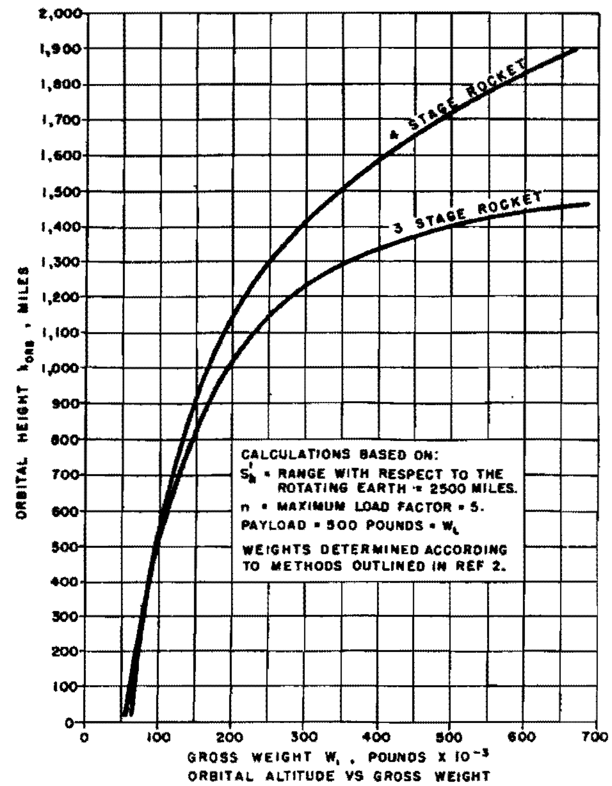
10. The Accuracy With Which The Orbital Conditions Must Be Established

Since it probably will not be possible in practice to attain exactly the circular orbital conditions which are specified, it becomes necessary to investigate the orbit which results if the specified circular orbital conditions are not exactly satisfied. If the flight values at the end of the trajectory differ from those which are specified in order to establish the desired circular orbit, the rocket will then, in general, establish an elliptical orbit. Although a circular orbit is the desired one, an elliptical orbit will be acceptable under certain conditions.



VARIATION OF GROSS WEIGHT WITH NUMBER OF STAGES FOR INDEPENDENTLY STAGED SATELLITE ROCKETS EMPLOYING DIFFERENT PROPELLANT SYSTEMS.

FIG. 18



INITIAL GROSS WEIGHT OF A HYDRAZINE - OXYGEN SATELLITE ROCKET AS A FUNCTION OF ORBITAL HEIGHT.

FIG. 19

From the drop per revolution which results (section 11) because of the atmospheric density values prevailing in the upper atmosphere[4], it is found desirable to establish the circular orbit at a height of 350 miles if a long duration of orbital motion is required. Since the minimum duration acceptable corresponds to an initial circular orbital height of about 250 miles, if an elliptical orbit is established it follows that the very least condition which will be acceptable is that the ellipse be such that its *minimum* height from the surface of the earth be 250 miles. Since it is safer to require, when an elliptical orbit is established instead of a circular one, that the height of the elliptical orbit never fall below about 300 miles, this value for the minimum height will also be considered in the following discussion. In view of these limitations on height for an elliptical orbit, it becomes necessary to determine the corresponding amount of error which may be allowed, for example, in the final values of velocity and angle of inclination at the end of the trajectory.

In answering these questions use will be made of the properties of elliptical motion derived in Appendix I (see Fig. 23) and these results will be used here without any additional definition or explanation. In Appendix I, relations (166), it was found that the maximum and minimum distances of the ellipse from the center of the earth, the points 1 and 2 in Fig. 23, have the values

$$r_{max} \doteq r_1 = a + c, \text{ where } \theta_1 = 0,\ \phi_1 = 0, \text{ and} \tag{118}$$

$$r_{min} = r_2 = a - c, \text{ where } \theta_2 = 0,\ \phi_2 = \pi. \tag{119}$$

It was also shown, Eq. (167), that the parameter μ has its minimum value at point 1 where

$$\mu_{min} = \mu_1 = 1 - \frac{c}{a}, \tag{120}$$

and its maximum value at point 2 where

$$\mu_{max} = \mu_2 = 1 + \frac{c}{a}. \tag{121}$$

It was shown further, Eq. (170), that at the points 3 and 4, Fig. 23, where the minor axis meets the ellipse, and where $r = r_3 = r_4 = \sqrt{b^2 + c^2} = a$, the parameter μ has the value $\mu = \mu_3 = \mu_4 = 1$. The maximum angle of inclination was found to occur at the point 3, having the value, Eq. (168),

$$\theta_{max} = \theta_3 = \tan^{-1} \frac{c}{b}, \tag{122}$$

while the minimum angle occurs at the point 4 and has the value

$$\theta_{min} = \theta_4 = -\tan^{-1} \frac{c}{b}. \tag{123}$$

Thus, $\theta_{min} = -\theta_{max}$, or the extreme value of θ is always given by

$$|\theta_{ext}| = \tan^{-1} \frac{c}{b}. \tag{124}$$

Since $\sqrt{b^2 + c^2} = a$, it follows that this may also be written

$$|\theta_{ext}| = \sin^{-1} \frac{c}{a}. \tag{125}$$

Suppose now that the required orbital height h_{orb} and velocity v_{orb} (the subscript req used previously to denote required values is no longer necessary here) for the circular orbit are obtained at the end of the trajectory, but that the angle is wrong. When the height and velocity stand in correct relation for a circular orbit, it follows from Eq. (56) that $\mu = 1$ and therefore, referred to an ellipse, the rocket must be situated at a point such as 3 or 4 where $r = \sqrt{b^2 + c^2} = a$. Thus the orbital distance at which the orbit is established in this case is $r_{orb} = a$. Since the correct value for the angle is $\theta = 0°$ and since the actual value is $|\theta| = \sin^{-1} c/a$, the error in angle, which will be denoted by $|\Delta\theta|$, is simply $|\Delta\theta| = \sin^{-1} c/a$, or

$$\sin |\Delta\theta| = \frac{c}{a}, \tag{126}$$

where $a = r_{orb}$. Since an elliptical orbit, Fig. 23, will be established instead of a circular one if there is an error in angle, and since this will be established at the points 3 or 4, where $r = a = r_{orb}$, it follows that the greatest decrease in height during a traverse of the elliptical orbit will occur at the point 2, and will therefore be of amount $r_{orb} - (a - c) = c$. Letting $\Delta h = c$ denote the decrease in height which occurs during a traverse of the elliptical orbit, and replacing, for small angles, the sine by the angle, it follows from Eq. (126) that

$$|\Delta\theta| = \frac{\Delta h}{r_{orb}}. \tag{127}$$

Using $h_{orb} = r_{orb} - R = 350$ miles, it is found that $|\Delta\theta| = 1.3°$ for $\Delta h = 100$ miles and $0.6°$ for $\Delta h = 50$ miles. Thus if the minimum allowable height above the earth's surface is 300 miles, the angle of inclination at the end of the trajectory must be controlled with an accuracy of about one-half of a degree.

Having discussed the effect of an error $|\Delta\theta|$ in the angle, we now assume that the proper values of height and angle, h_{orb} and $\theta_{orb} = 0$, to establish a circular orbit exist at the end of the trajectory but that the velocity is too low. In this case also an elliptical orbit will be established, and since $\theta_F = \theta_{orb} = 0$, the ellipse will be established either at the point 1 or 2. Since the velocity is specified as being too low, it follows from Eq. (56) that μ must be less than unity and therefore that the elliptical orbit must start at the point 1 where $\mu = \mu_1 = 1 - c/a$ and where $r_{orb} = r_1 = a + c$. It then follows in this case that $\Delta h = r_1 - (a - c) = 2c$. Since $\mu_1 = 1 - c/a$, the quantity μ_1 may be written

$$\mu_1 = 1 - \frac{\Delta h}{2r_{orb} - \Delta h},$$

and since r_{orb} is of the order of 4300 miles while Δh is only 50 to 100 miles, it is permissible to neglect Δh compared to $2r_{orb}$ and to use $\mu_1 = 1 - \Delta h/2r_{orb}$, or

$$1 - \mu_1 = \frac{\Delta h}{2r_{orb}}. \tag{128}$$

If $-\Delta v$ represents the error in the velocity v_{orb}, required for the circular orbit, the velocity v_1 at the point 1 is $v_1 = v_{orb} - \Delta v$, and from the definition $\mu = v^2/gr$ and from relation (56) it follows that

$$\mu_1 = \frac{(v_{orb} - \Delta v)^2}{gr_{orb}} = \frac{(v_{orb} - \Delta v)^2}{v_{orb}^2} = \left(1 - \frac{\Delta v}{v_{orb}}\right)^2.$$

Since v_{orb} is of the order of 25000 ft/sec, the quantity $(\Delta v/v_{orb})^2$ appearing in the expansion will be a small higher order term which may be safely neglected giving

$$1 - \mu_1 = 2\,\frac{\Delta v}{v_{orb}}. \tag{129}$$

A comparison of Eqs. (128) and (129) results in the relation

$$\frac{\Delta v}{v_{orb}} = \frac{\Delta h}{4 r_{orb}} . \tag{130}$$

Using h_{orb} = 350 miles it is found that an error in velocity of 0.6% gives a value Δh of 100 miles, while an error of 0.3% gives 50 miles.

When the trajectory values v_i, θ_i, r_i at the beginning of coasting differ, because of error or inaccuracy, from those which are required in order to satisfy specified orbital conditions, an elliptical orbit will be established rather than a circular one. Errors in the values v_i, θ_i, r_i may be partially compensated for by adjusting the coasting height interval Δh_c in such a way that the elliptical orbit which is established has its minimum distance from the earth $r_2 = (a - c)$ as large as possible. Since τ is always taken as large as possible consistent with range restrictions, it is found that this leads to values of the angle θ_f, at the end of coasting, which are practically zero, of the order of a few tenths of a degree (see Fig. 16 for example). The best way of adjusting Δh_c to compensate for existing errors in the initial coasting conditions is to adjust the duration of coasting so that θ_f has a value as close to zero as possible. The proof of this statement is contained in Appendix I. A method is discussed in the Communication Report[12] for obtaining the condition $\theta_F = 0$ by adjusting the duration of coasting on the basis of measurements made during flight of the conditions at the beginning of coasting.

Table 1

ERROR IN VELOCITY OR ANGLE AT THE BEGINNING OF COASTING TO GIVE SPECIFIED VALUES OF $r_{orb} - r_{2\,max}$

$r_{orb} - r_{2\,max}$	$\frac{v_i - v_{io}}{v_{io}}$	$\theta_i - \theta_{io}$, degrees	Δv_F, ft/sec
50 miles	0.0036 0.	0. 0.83°	66 115
100 miles	0.0072 0.	0. 1.62°	132 230

v_{io} and θ_{io} are the values of velocity and angle at the beginning of coasting when there exists no error of any kind.

Table 1 gives the error, at the end of coasting, in either the velocity v_f (when the error in angle is zero) or the angle θ_f (when the error in velocity is zero) which results in the values of 50 and 100 miles for the difference $r_{orb} - r_{2\,max}$. The notation $r_{2\,max}$ refers to the elliptical orbit which has its minimum distance r_2 as large as possible. The figures in the table show that the path control apparatus must be designed to control the flight path with an accuracy of either 1% in velocity or 1° in angle. The table also shows the increase in velocity which is required, over and above that of the final burst, in order to establish a circular orbit. This last increase in velocity could be accomplished if a slight reserve of propellants is available.

As far as stability and control of the rocket are concerned, the aerodynamic stability is to be accomplished by means of fins[6] and the dynamic control[13] is to be brought about by the use of rocket control motors[14].

In order to insure that the attitude (i.e. the angle of tilt α) of the rocket, when it starts in the final burst, is the desired one with the nose of the rocket always pointing in the direction of motion, it may be necessary to allow for a slight change in the shape of the final part of the trajectory. This necessity arises because, unless the control motors are operating, the attitude of the rocket will change by about 30° during coasting. If the control motors are used during coasting, the coasting period becomes essentially analagous to a very long burning period stage having a value of n of about 0.08. Although this would necessitate a slight compensatory change in v_i and v_f, the net effect on required gross weight to reach a 350 mile orbit is quite negligible.

11. Stability of the Orbital Motion and Duration of Flight After the Orbit is Established

Once the satellite body is established on its circular orbit, it is important to know the effect on the orbital motion of the drag resulting from the extremely small density of the rarefied atmosphere[4] existing at the orbital height. Although the drag itself is extremely small, since it is continually present while the rocket is revolving about the earth, the integrated effect over large time intervals results eventually in a decrease in orbital height such that the rocket finally reaches the surface of the earth. Thus, because of the delicate balance which exists between the gravity and centripetal forces when the motion is orbital, the stability of such motion is quite sensitive to the presence of any dissipative (drag) forces. Assuming the initial orbital height to be 350 miles, we shall investigate the time and number of revolutions about the earth required before the rocket drops to a height of 100 miles. This height interval, 350 to 100 miles, is chosen for discussion since conditions remain practically orbital in this range, and the decrease in height per revolution of the satellite in its orbit may be determined from a particularly simple formula.

Assuming that the satellite has been initially established on a circular orbit (at a height of 350 miles for example), the problem is to integrate the equations of motion of a free body moving in a central force field in which dissipative (drag) forces are present, and thus to find the variation of the motion with time. Since it is not possible, in general, to integrate these equations of motion in closed form, approximate methods must be employed. One such method which is especially appropriate to the present problem has been given by Chien[15]. This analysis is given in Appendix II. Referred to a fixed non-rotating system of coordinates with origin at the center of the earth, the approximate integration of the equations of motion in polar coordinates as derived in Appendix II gives the result

$$\Delta v_1 = \frac{1}{\omega_0}(\omega_0 t - 2\sin\omega_0 t)$$

$$\Delta r_1 = \frac{2}{\omega_0^2}(\sin\omega_0 t - \omega_0 t) \tag{131}$$

$$\Delta\omega_1 = \frac{1}{r_0\omega_0}(3\omega_0 t - 4\sin\omega_0 t) \ ,$$

where r_0, v_0, and ω_0 are the initial equilibrium conditions at the beginning of the circular orbital motion, and Δr_1, Δv_1, $\Delta\omega_1$ are the deviations or perturbations from these equilibrium conditions resulting from the effects of the atmospheric drag D.

Writing $D/m = g_s D/W$, the variation in height $r - r_0$ is expressed by $r - r_0 = g_s\,(D/W)\,\Delta r_1$, or

$$r - r_0 = \frac{2}{\omega_0^2}(\sin\omega_0 t - \omega_0 t)\ g_s \frac{D}{W} . \tag{132}$$

The corresponding variation in velocity is

$$v - v_0 = \frac{1}{\omega_0}(\omega_0 t - 2\sin\omega_0 t)\ g_s \frac{D}{W} . \tag{133}$$

Although the values of v and r computed from these equations show some variation over an orbital period, it is found that the final values at the end of the revolution are almost exactly the same as the values at the beginning, and therefore the net effect of drag during only one revolution is very small. However, the integrated effect of drag over a large number of revolutions becomes appreciable as will be shown by the following analysis.

For one complete revolution of the satellite, the change in height Δr is determined by

$$\Delta r = g_s \frac{D}{W}\Delta r_1 \Bigg]_0^{\frac{2\pi}{\omega_0}} = -\,g_s \frac{D}{W}\frac{4\pi}{\omega_0^2} . \tag{133a}$$

Making use of the relation

$$\frac{v_0^2}{r_0} = r_0\omega_0^2 = g_0, \quad \text{where } g_0 = g_R\left(\frac{R}{r_0}\right)^2 ,$$

Eq. (133a) may be written

$$\Delta r = -4\pi \frac{g_s}{g_0} r_0 \frac{D}{W}. \tag{134}$$

Using the appropriate value for D based on the designed body size, it is found, for example, that at a height of 350 miles, $\Delta r = -34$ feet.

Let N_r denote the total number of revolutions (referred to fixed space and not to the rotating earth) which the satellite must make before the height has decreased to a certain value h_L, say 100 miles. Since N_r will be a relatively large number, the decrease in height per revolution Δr may be replaced by the derivative dr/dN_r giving the differential equation

$$dr = -4\pi \frac{g_s}{g_0} \frac{D}{W} r dN_r \, . \tag{135}$$

Using $D = C_D \frac{\rho}{2} Av^2$ with $C_D = 2$ as discussed in Ref. 6, this becomes

$$dN_r = -\frac{g_0}{g_s} \frac{W}{2\pi C_D \, Av^2 r} \frac{dr}{\rho(r)} \, . \tag{136}$$

Since for relatively small changes in height the percentage variation in r itself will be small, the density $\rho(r)$ will be the only quantity in this expression which varies appreciably with r, and to a fairly close approximation we may use

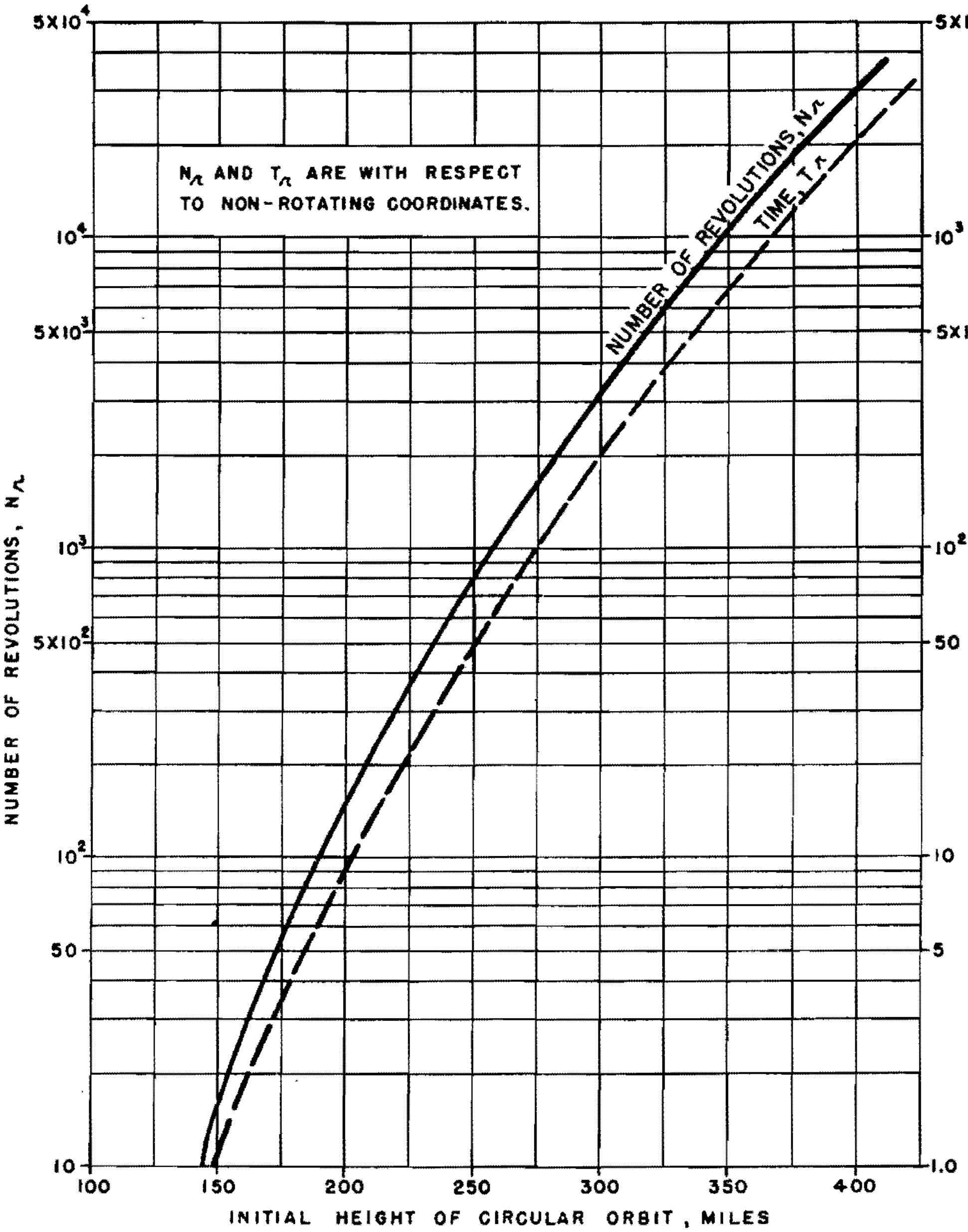
$$N_r = -\frac{W}{2\pi A} \frac{g_0}{g_s} \sum_k \frac{1}{\left(\overline{C_D r v^2}\right)_k} \int_{r_{2k}}^{r_{1k}} \frac{dr}{\rho(r)} \quad , \tag{137}$$

where the bar indicates the mean value appropriate to the height interval $(r_{2k} - r_{1k})$, r_{2k} is the upper level of the kth height interval, and r_{1k} the lower level. The total height interval covered by the integration is $\sum_k (r_{2k} - r_{1k}) = r_0 - r_L$.

Since $d(\log \rho)/dr$ is approximately constant if the change in height is not too large, a variation in density of the form

$$\rho_k = \rho_{2k} \exp\,(r_{2k} - r_k)\, K_k \quad , \tag{138}$$

is used, where ρ_{2k} is the density at an upper level at distance r_{2k}, ρ_k is the density at any lower level r_k within the kth interval, and K_k is a constant which determines the density variation within the kth height interval and is different for each height interval chosen. In practice it is found satisfactory over ranges in height from 350 down to 100 miles to perform the integration of Eq. (137) by using the density variation (K_k) based on 50 mile intervals. The density values used are those given in Ref. 4 for the atmosphere at the equator. Proceeding in this manner, the values given in Fig. 20 are obtained for the number of revolutions N_r required in order for the satellite to drop from an initial orbital height h_0 to a height $h_L = 100$ miles.



NUMBER OF REVOLUTIONS AND CORRESPONDING TIME REQUIRED FOR SATELLITE TO DROP TO A HEIGHT OF 100 MILES STARTING FROM VARIOUS INITIAL ORBITAL HEIGHTS.

FIG. 20

The period P of a revolution is

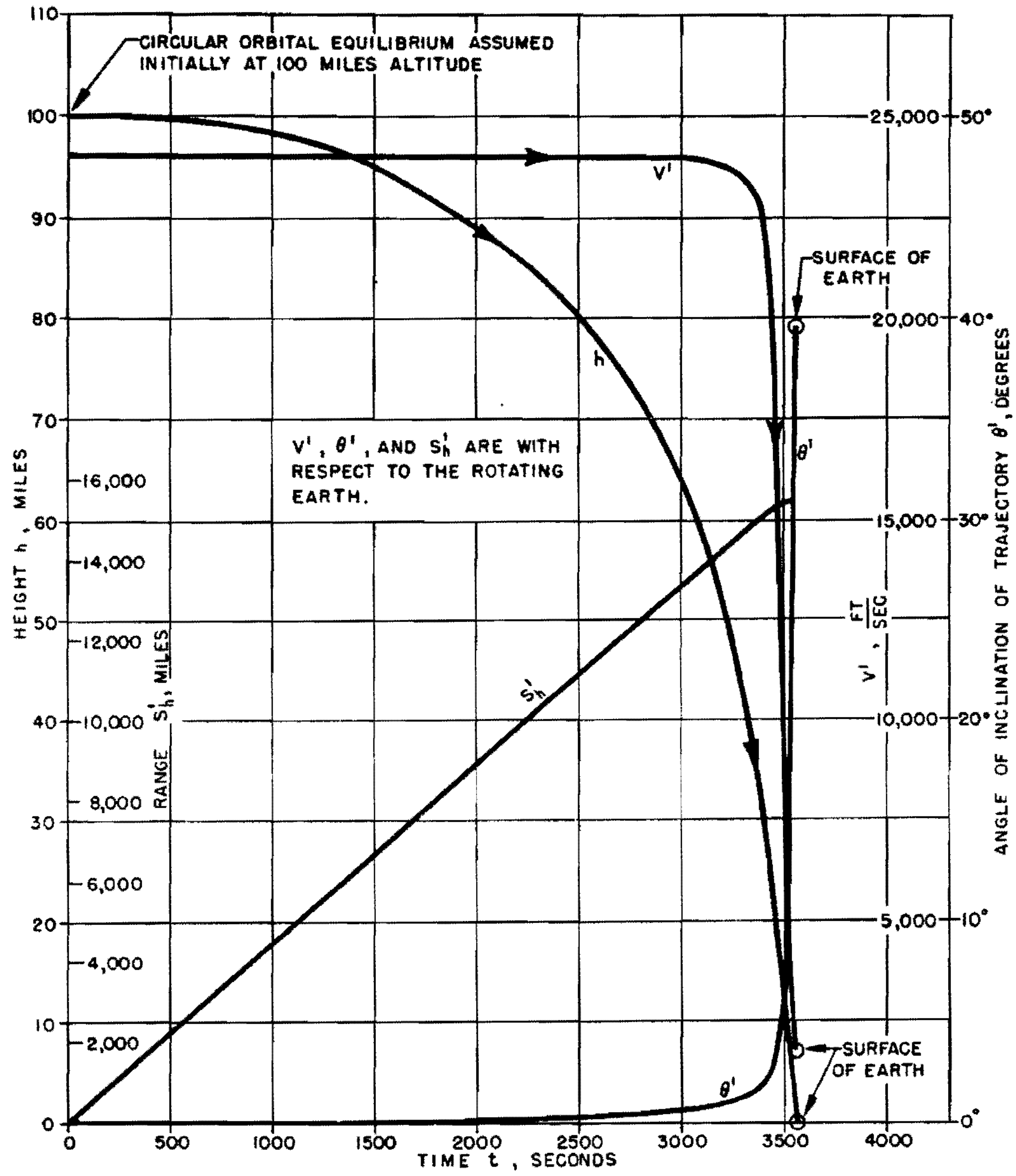
$$P = \frac{2\pi r}{v}, \tag{139}$$

and since r and v, and especially the ratio r/v, do not vary much in the range 350 to 100 miles, the value of P remains essentially constant. The time T_r required for the rocket to drop from 350 down to 100 miles height may therefore be computed from

$$T_r = 2\pi\left(\overline{\frac{r}{v}}\right)N_r \quad . \tag{140}$$

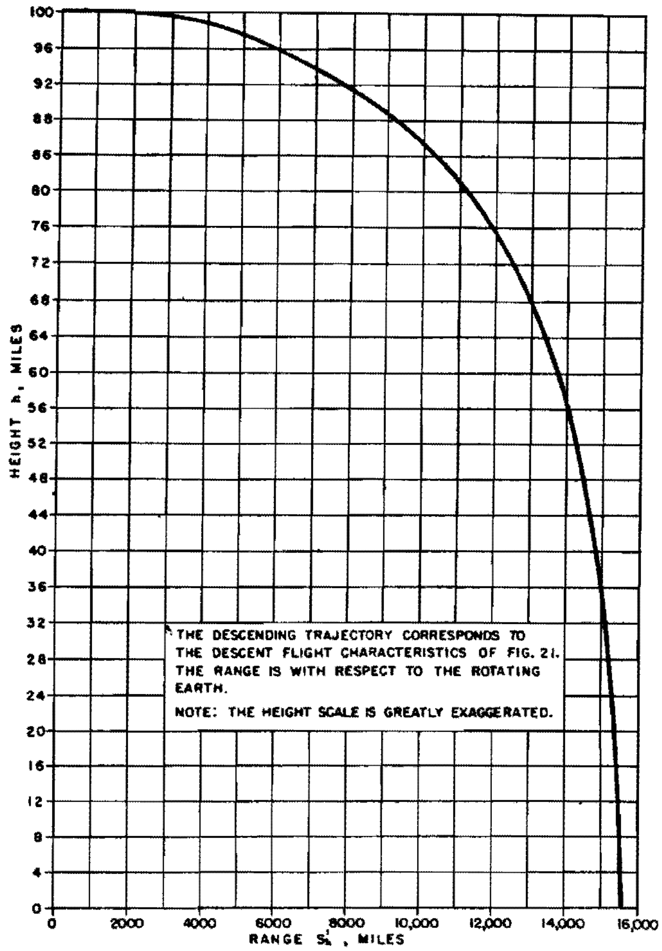
The values of T_r corresponding to those of N_r are also shown in Fig. 20. This figure shows that when the orbit is established at a height of 350 miles, the satellite will remain aloft for approximately two years. It also shows that a period of about 6 months must elapse before the satellite drops to the earth starting from a height of 300 miles, and 46 days when the satellite starts from a height of 250 miles. Starting at an altitude of 100 miles it is found that the descent of the satellite is so rapid that it reaches the earth in less than one revolution (less than 1½ hours), so that this portion of the descent is negligible as far as duration is concerned. It is precisely these results which have led to the choice of 350 miles as the desired orbital height. For, since a duration of 40 days for the 250 mile height is the least amount acceptable, it is found necessary to specify a 350 mile height in order not to violate the 250 minimum height limitation if an elliptical orbit is established instead of a circular one due to inaccuracies, etc., as discussed previously.

The accuracy and limitations of the method of calculation presented above for the decrease in altitude of the satellite may be investigated by determining the second approximation values, Δv_2, Δr_2, $\Delta\omega_2$ occurring in Eq. (176), Appendix II. This has been done and the results show that the calculations presented here are valid down to heights of about 100 miles. The analysis of the second approximation is quite involved and will be presented later in a separate paper. For heights lower than 100 miles, the increase in density with decreasing height is so rapid, and the consequent variation with time of v, r, and θ becomes so large, that the method employed above is no longer applicable. In fact below about 75 miles the motion changes so rapidly that it becomes necessary to perform a numerical integration of the equations of motion in order to obtain a satisfactory solution. The results of these calculations show that once the satellite has descended to an altitude of 100 miles, it will then descend rapidly to the earth in less than one revolution. The accurate calculation (by numerical integration of the equations of motion) of the descent of the satellite flying stably from a height of 100 miles down to the earth has been carried out taking into account the variations of density(4) and drag coefficient(6), and the results are shown in Figs. 21, 22 and 22A. Fig. 21 contains the complete results of the calculations and shows the variation with time of height, velocity, angle of inclination of trajectory, and the range. The values given are with respect to an observer situated on the rotating earth. Although, strictly speaking, exact circular orbital equilibrium exists only at the 350 mile height where the circular orbit is established, it is found that even after the satellite has descended to 100 miles height, the velocity and angle differ by such a small amount from exact circular orbital conditions corresponding to this height that it is quite permissible to use exact equilibrium as the initial condition at 100 miles height.



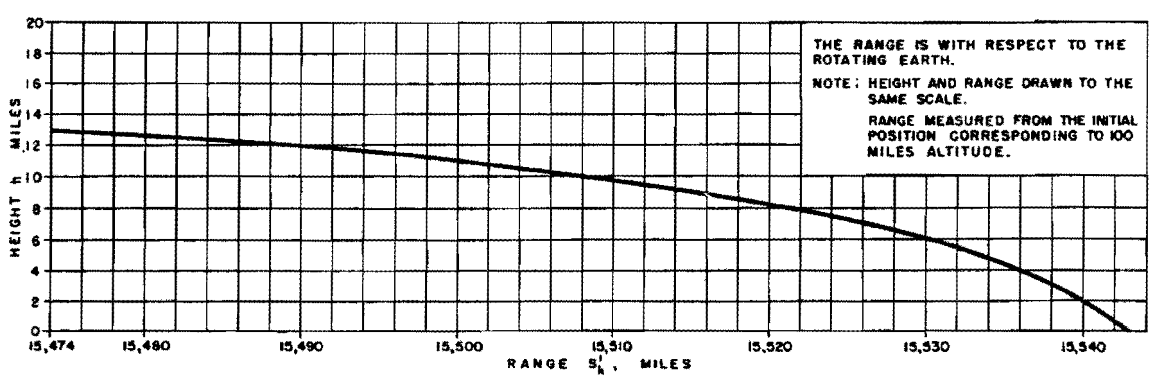
FLIGHT CHARACTERISTICS OF THE DESCENT OF SATELLITE FROM A HEIGHT OF 100 MILES DOWN TO THE SURFACE OF THE EARTH.

FIG. 21



DESCENDING TRAJECTORY, HEIGHT VS RANGE, OF THE SATELLITE FROM A HEIGHT OF 100 MILES DOWN TO THE SURFACE OF THE EARTH.

FIG. 22



FINAL PORTION OF THE DESCENDING TRAJECTORY OF THE SATELLITE SHOWN IN FIG. 22.

FIG. 22A

12. Conclusions

From the analysis and calculations which have been presented it is found that a rocket may be established on a satisfactory satellite orbit by using a three stage rocket with the hydrazine-oxygen propellant system. It is found that the minimum gross weight required for the three stage rocket is 84,400 pounds, which is about three times that of the German A4 rocket.

While it is apparent from Fig. 18 that the minimum gross weight required is much less when the hydrogen-oxygen or the hydrazine-fluorine propellant systems are used, the hydrazine-oxygen system has been chosen because of the much greater ease and safety in its handling and because of its more ready availability. A three stage rocket has been chosen since, for the propellants of interest, the gross weight is much less than for a two stage rocket. While the three and four stage rockets are found to have about the same gross weight, the three stage rocket is chosen because it will obviously be less complex from a fabrication standpoint.

For the three stage rocket it has been found that practically optimum conditions will be obtained when the burning is such that the rate of mass flow of propellants is constant during a burning period, and when n and ν have the same value for each burning period. Since the final optimum values of n lie in the range from 4 to 5, it seems likely that the corresponding accelerations might be just low enough to allow a man to ride in the rocket without too much risk of blacking out or of injury to internal organs.

An analysis of orbital stability on the basis of the vertical distribution of atmospheric density found in Ref. 4 leads to the conclusion that the orbit of the satellite rocket should be established at a height of 350 miles if satisfactory duration is to be guaranteed, especially in view of the fact that slight errors in the trajectory will cause an elliptical orbit to be established instead of a circular one. On the basis of a required orbital height of 350 miles, extensive calculations have been made to determine the optimum trajectory, this being the trajectory requiring the least gross weight necessary to establish the orbit.

The determination of the best trajectory is subject to several restrictions of a practical nature, one of these being the range. In order to satisfy the requirements of communication with, and tracking of, the satellite rocket, it is found necessary to restrict the range of the trajectory so that it never exceeds 2500 miles. Another restriction or condition which the trajectory must satisfy is one concerning the temperature of the metal (stainless steel) skin of the rocket. Unless care is exercised, the extremely high speed motion of the rocket through the atmosphere might easily lead to sufficiently high skin temperatures that melting would result. Accordingly, the trajectory has been required to satisfy the condition that the maximum skin temperature must not exceed 2000°R, and this amount is allowed for only a short time[6]. A third condition which the trajectory has been made to satisfy is that the control moments required of the servomechanism do not exceed the moment producing limits of such mechanisms. The optimum or best trajectory consistent with these various limitations is shown in Fig. 15. The outstanding features in connection with this optimum trajectory are (1) the small amounts of tilt employed and the positioning of the maximum tilt early in the trajectory and (2) the use of long duration coasting positioned near the end of the trajectory.

APPENDIX I.

PROPERTIES OF ELLIPTICAL MOTION

a. Integration of the Equations of Motion for Coasting--The Equations of Energy and Momentum

It was pointed out in section 5 of Part II that the motion of the satellite rocket is such that the coasting part of the trajectory is always a portion of an ellipse with the center of the earth situated at one focus. Over the coasting part of the trajectory the rocket moves as a free body in the central force field of gravity for which the differential equations of motion are

$$\frac{dv}{dt} = -g \sin\theta , \tag{141}$$

$$v\frac{d\theta}{dt} = -g\cos\theta + \frac{v^2}{r}\cos\theta , \quad \text{and} \tag{142}$$

[illegible] (143)

Letting s denote distance along the trajectory, we may write

$$\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds} = v\frac{dv}{dr}\frac{dr}{ds} = v\frac{dv}{dr}\sin\theta = -g\sin\theta ,$$

where $dr/ds = \sin\theta$, since θ is the angle of inclination referred to the instantaneous horizontal. Hence,

$$\frac{d}{dr}\left(\frac{v^2}{2}\right) = -g . \tag{144}$$

Using the gravity relation $g = g_R(R/r)^2$ and denoting conditions at the beginning of the coasting motion by the subscript i, the integration of Eq. (144) gives the total energy equation

$$v^2 - v_i^2 = 2g_R R^2\left(\frac{1}{r} - \frac{1}{r_i}\right) = +2g_R\left(\frac{R}{r}\right)^2 r - 2g_R\left(\frac{R}{r_i}\right)^2 r_i$$

$$= -2g_i r_i + 2gr . \tag{145}$$

Since, as explained below, the total energy $2E$ of the satellite rocket will always be negative, by putting $2E = -U^2$ (where U is real) the energy equation may be expressed by

$$v^2 - 2gr = v_i^2 - 2g_i r_i = 2E = -U^2 . \tag{146}$$

If further, the absolute gravity relation $g = g_R (R/r)^2$ is introduced, this becomes

$$v^2 - 2g_R \frac{R^2}{r} = v_i^2 - 2g_R \frac{R^2}{r_i} = -U^2 . \tag{147}$$

Considering Eq. (142) we may write

$$v \frac{d\theta}{dt} = v^2 \left(\frac{d\theta}{ds}\right) = v^2 \left(\frac{d\theta}{dr}\right) \sin \theta$$

and Eq. (142) becomes

$$v^2 \left(\frac{d\theta}{dr}\right) \sin \theta = -g \cos \theta + \frac{v^2}{r} \cos \theta , \quad \text{or}$$

$$\tan \theta \, d\theta = \frac{-g + \frac{v^2}{r}}{v^2} dr = -\frac{g}{v^2} dr + \frac{dr}{r} , \tag{148}$$

Substituting v^2 from Eq. (145) and integrating yields

$$\log \frac{\cos \theta}{\cos \theta_i} + \frac{1}{2} \log \left(\frac{v_i^2 - 2g_i r_i + 2gr}{v_i^2} \right) + \log \frac{r}{r_i} .$$

Using relation (145) again, this gives the angular momentum equation

$$rv \cos \theta = r_i v_i \cos \theta_i = \text{constant} = M . \tag{149}$$

b. Special Properties of an Elliptical Orbit

The integrals (145) and (149) of the motion of a free body are, of course, quite general and apply to a flight path of any shape, the elliptical property of the motion having not yet been introduced. However, when the total energy constant $2E$ is specified as negative, this at once requires that the motion be elliptical. This is readily shown by introducing the polar angle ϕ, in terms of which the angular momentum M is expressed by

$$M = r^2 \frac{d\phi}{dt} = r^2 \frac{d\phi}{dr}\frac{dr}{dt}. \tag{150}$$

The energy relation may then be written

$$2E = \left(\frac{dr}{dt}\right)^2 + \left(r\,\frac{d\phi}{dt}\right)^2 - 2g_R\frac{R^2}{r} = \text{constant, or} \tag{151}$$

$$2E + 2g_R\frac{R^2}{r} = \frac{M^2}{r^2} + \frac{M^2}{r^4\left(\dfrac{d\phi}{dr}\right)^2}. \tag{152}$$

The total energy $2E$ is zero when the body is at rest at infinity, and as seen from Eq. (146), will always be negative provided v is less than the escape velocity. Since for the satellite rocket, v will always be less than the escape velocity, it follows that the values of $2E$ considered here must always be negative, and it is therefore permissible to use $2E = -U^2$, where it will be noted that U has the dimensions of velocity.

Eq. (152) is readily integrated to give

$$\frac{1}{r} = g_R\frac{R^2}{M^2}\left[1 - \sqrt{1 + \frac{2EM^2}{(g_R R^2)^2}}\ \cos(\phi - \delta)\right], \tag{153}$$

where δ is a constant of integration. This is the equation in polar coordinates of a conic section referred to one focus as origin. The constant of integration δ determines the position of the apse-line which is defined as the line on which $dr/d\phi$ changes sign[11] and therefore the line on which r attains its maximum or minimum value. Choosing this as the line along which $\phi = 0$ by putting $\delta = 0$, we have

$$\frac{1}{r} = g_R\frac{R^2}{M^2}\left[1 - \sqrt{1 + \frac{2EM^2}{(g_R R^2)^2}}\ \cos\phi\right]. \tag{154}$$

That this equation represents an ellipse may be seen, for example, by direct comparison with the usual equation of the ellipse in polar coordinates when referred to one focus, as derived by the methods of analytic geometry. When the origin is located at the left focus F_1, Fig. 23, the equation for an ellipse in polar coordinates is

$$\frac{1}{r} = \frac{a}{b^2}(1 - e\cos\phi), \tag{155}$$

where a is the semi-major axis, b is the semi-minor axis, and e is the eccentricity defined by $e = \sqrt{1 - b^2/a^2}$. Comparison with Eq. (154) shows that

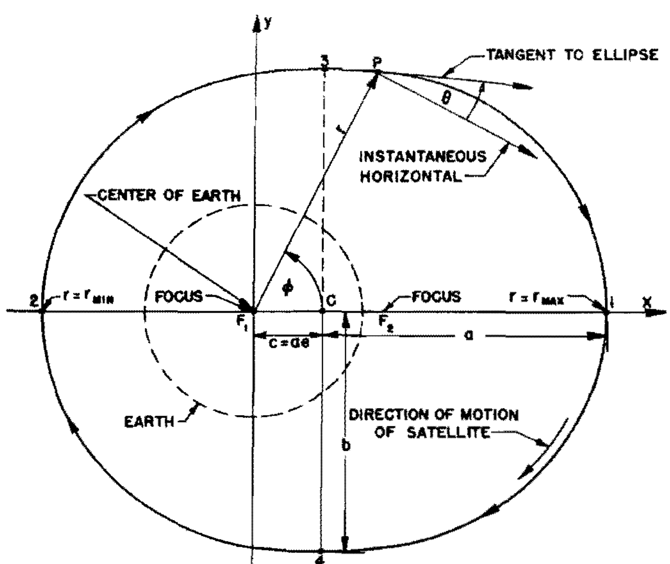
$$e^2 = 1 + \frac{2EM^2}{(g_R R^2)^2} = 1 - \frac{b^2}{a^2}, \quad \text{and}$$

$$g_R \frac{R^2}{M^2} = \frac{a}{b^2}.$$

Hence, using $2E = -U^2$,

$$a = -\frac{g_R R^2}{2E} = \frac{g_R R^2}{U^2}, \quad b = \frac{M}{U}, \quad \text{and} \quad e = \sqrt{1 - \frac{U^2 M^2}{(g_R R^2)^2}}, \tag{156}$$

which shows that $2E$ must be negative for elliptical motion. Thus, Eq. (154) represents an ellipse such that the origin, which is at the center of the earth, is situated at the left focus as shown in Fig. 23.



C = CENTER OF ELLIPSE
a = SEMI-MAJOR AXIS
b = SEMI-MINOR AXIS

$c = F_1C = ae = \sqrt{a^2 - b^2}$

e = ECCENTRICITY = $\frac{\sqrt{a^2-b^2}}{a}$

SCHEMATIC DIAGRAM TO ILLUSTRATE ELLIPTICAL MOTION

THE DIRECTION OF MOTION INDICATED CORRESPONDS TO THAT OF THE SATELLITE WHEN VIEWED FROM THE SOUTH

FIG. 23

Since $d\phi/dt = M/r^2$, Eq. (151) may be written

$$2E = U^2 = \left(\frac{dr}{dt}\right)^2 + \frac{M}{r^2} - 2g_R\frac{R^2}{r}\ , \text{ or}$$

$$\frac{dr}{dt} = \frac{U}{r}\sqrt{2ra - r^2 - b^2} = \frac{U}{r}\sqrt{c^2 - (r-a)^2}\ , \tag{157}$$

where relations (156) have been used and where

$$c = ae = \sqrt{a^2 - b^2}\ . \tag{158}$$

The negative radical sign, omitted in (157) refers to the third and fourth quadrants. The integral of this equation is

$$tU = -\left[a\cos^{-1}\frac{r-a}{c} + \sqrt{c^2 - (r-a)^2}\right] + \text{constant.} \tag{159}$$

Since $dr/dt = v\sin\theta$, it follows from (157) that

$$\sqrt{c^2 - (r-a)^2} = \frac{rv\sin\theta}{U} = b\tan\theta, \quad \text{and}$$

$$\frac{r-a}{c} = \pm\sqrt{1 - \left(\frac{b}{c}\tan\theta\right)^2}\ , \tag{160}$$

where relations (149) and (156) have been used. The positive sign of the radical is used in the first quadrant and the negative sign in the second quandrant. Hence, Eq. (159) may now be written in the form

$$t = -\frac{1}{U}\left[a\sin^{-1}\left(\frac{b}{c}\tan\theta\right) + b\tan\theta\right] + \text{constant,} \tag{161}$$

which may be used to find the time of transit between two points i and f on an elliptical path by forming the difference $t_f - t_i$.

From (154) or (155) one obtains

$$\tan\phi = \frac{b}{ar - b^2}\sqrt{c^2 - (r-a)^2}\ ,$$

which, by use of (160), becomes

$$\tan\phi = \frac{b^2}{ar - b^2}\tan\theta\ . \tag{162}$$

Introducing the parameter μ defined by

$$\mu = \frac{v^2}{gr} = \frac{v^2 r}{g_R R^2}, \tag{163}$$

it follows from relations (156) that

$$\frac{b^2}{a} = \frac{M^2}{g_R R^2} = \mu r \cos^2 \theta, \text{ or, } r = \frac{b^2}{a\mu \cos^2 \theta}, \tag{164}$$

and if this value for r be substituted in Eq. (160), the resulting expression is

$$\phi = \tan^{-1} \left[\frac{\mu \tan \theta}{\tan^2 \theta + (1 - \mu)} \right]. \tag{165}$$

By using this equation, the range $S_h = R(\phi_i - \phi_f)$ may be computed from a knowledge of the initial values θ_i, μ_i and the final values θ_f, μ_f corresponding to a portion of an elliptical path. The corresponding time of transit is obtained from Eq. (161).

Further remarks which follow readily from the preceding analysis, may be made concerning the ellipse. In the ellipse of Fig. 23 consider the points 1,2,3, and 4 situated at the ends of the major and minor axes. From Eq. (154) or (155) it is evident that r is a maximum when $\phi = 0$ (the point 1), and a minimum where $\phi = \pi$ (the point 2). Thus at the point 1,

$$r_1 = r_{max} = a + c, \quad \phi_1 = 0, \quad \theta_1 = 0,$$

and at the point 2, (166)

$$r_2 = r_{min} = a - c, \quad \phi_2 = \pi, \quad \theta_2 = 0.$$

From the energy equation (147) it is seen that v is a minimum when r is a maximum, and from (163) it then follows that μ is a minimum when r is a maximum, and vice versa. Thus, using relations (158) and (164), it is found that at the point 1 where $r_1 = r_{max}$,

$$\mu_1 = \mu_{min} = 1 - \frac{c}{a},$$

and at the point 2 where $r_2 = r_{min}$, (167)

$$\mu_2 = \mu_{max} = 1 + \frac{c}{a}.$$

The points 3 and 4 also have some special properties. At these two points r has the value $r = r_3 = r_4 = \sqrt{b^2 + c^2} = a$. From Eq. (160) we have $\tan \theta = 1/b \sqrt{c^2 - (a-r)^2}$ from which it is readily evident that θ will have an extreme value when $r = a$. Thus, θ will be either a maximum or minimum at the points 3 and 4. Since θ is positive in the counterclockwise sense measured from the instantaneous horizontal PQ, Fig. 23, to the tangent to the ellipse, it is evident that θ will be a maximum at point 3 and a minimum (negative) at point 4. Hence, when $r = a$,

$$
\begin{aligned}
\theta_{max} &= \theta_3 = \tan^{-1}\frac{c}{b} \text{ at point 3, and} \\
\theta_{min} &= \theta_4 = \tan^{-1}\frac{c}{b} \text{ at point 4.}
\end{aligned}
\tag{168}
$$

Thus $\theta_{min} = -\theta_{max}$, and the extreme value of θ is always given by

$$|\theta_{ext}| = \tan^{-1}\frac{c}{b},$$

which, by use of $a = \sqrt{b^2 + c^2}$, may also be written in the form

$$|\theta_{ext}| = \sin^{-1}\frac{c}{a} = \cos^{-1}\frac{b}{a}. \tag{169}$$

Using $r = a$ and $\cos\theta = b/a$, it is found immediately from (162) that

$$\mu = \mu_3 = \mu_4 = 1, \tag{170}$$

at points 3 and 4.

c. Adjustment of Coasting to Compensate for Errors Present at the Beginning of Coasting

In section 10 of Part II it was pointed out that the best way of adjusting Δh_c to compensate for existing errors present in the initial coasting conditions is to adjust the duration of coasting so that θ_f (the angle of inclination at the end of coasting) has a value as close to zero as possible. The reasoning upon which this statement is based is contained in the following discussion.

When the flight conditions at the beginning of coasting (indicated by subscript i) are not exactly the prescribed ones owing to the presence of errors in v_i, r_i, and θ_i, it is important to know how the duration of coasting should be adjusted during flight so that the trajectory values v_f, r_f, θ_f, at the end of the trajectory will correspond to the best orbital condition it is possible to attain consistent with the errors present at the beginning of coasting. With given fixed values for Δv_e, Δr_e, $\Delta\theta_e$, and also for v_i, r_i, θ_i, then when Δh_c is specified, the final trajectory values v_F, r_F, θ_F, are determined, and therefore the elliptical orbit is determined. Using the results derived in section b of this appendix, the closest approach to the earth r_{min} of the elliptical orbit may be expressed in the form

$$r_2 = r_{min} = a - c = \frac{g_R R^2}{U_F^2} - \sqrt{\left(\frac{g_R R^2}{U_F^2}\right)^2 - \left(\frac{v_F r_F \cos\theta_F}{U_F}\right)^2}. \tag{171}$$

We now seek the value of either Δh_c or θ_F which gives the maximum value for r_{min}. Using Δh_c, this maximum condition is determined by an examination of the expression

$$\frac{dr_2}{d(\Delta h_c)} = \frac{1}{U_F^2 g_R R^2 e_F} \left\{ \frac{g_R R^2 (v_F r_F^2 - v_f r_f^2)}{v_f r_f^2 r_F^2} \left[2 g_R R^2 r_2 - (v_F r_F \cos \theta_F)^2 \right] \right.$$

$$+ U_F^2 r_F v_F \cos \theta_F \left[- \frac{g_R R^2 r_F \cos \theta_F}{v_f r_f^2} + v_F \cos \theta_F \right.$$

$$\left. \left. + \frac{v_F r_F (g_R R^2 - v_f^2 r_f)(\cos \theta_F \sin \theta_f - \sin \Delta\theta_e)}{v_f^2 r_f^2 \sin \theta_f} \right] \right\} . \qquad (172)$$

Putting $\theta_F = 0$ in this equation, it is found that

$$\left. \frac{dr_2}{d(\Delta h_c)} \right]_{\theta_F = 0} = - \left[1 + \frac{4 (v_F r_F^2 - v_f r_f^2)}{v_f r_f^2 (1 + e_F)^2} \right] ,$$

which shows that $\left. \frac{dr_2}{d(\Delta h_c)} \right]_{\theta_F = 0} < 0$.

The values of $\Delta\theta_e$ of interest lie in the range $|\Delta\theta_e| \leqq 0.4°$ corresponding to values of $\tau \geqq 0.70$. Using $\theta_F = 0.01°$ in Eq. (172) it is found that

$$\left. \frac{dr_2}{d(\Delta h_c)} \right]_{\theta_F = 0.01°} > 0.$$

Hence the derivative, Eq. (172), has its zero value, and therefore r_2 has its maximum value, within the narrow limits $0° < \theta_F < 0.01°$. Thus for all practical purposes the value $\theta_F = 0$ corresponds to the maximum value for r_2. Since the optimum trajectory calculations show that the $\Delta\theta_e$ required is of the order of 0.3°, it is evident in view of the limitations in accuracy of communication that $\Delta\theta_e$ may also be considered zero for all practical purposes. It is concluded therefore that when errors are are present at the beginning of coasting, it is best to allow the coasting to continue until the angle θ_f is practically zero. If the accuracy of communication were great enough, the optimum requirement would be that $\theta_f = \Delta\theta_e$.

APPENDIX II.

THE EFFECT OF DRAG FORCES ON THE STABILITY OF CIRCULAR ORBITAL MOTION

The problem considered here is that of the motion of a free body moving in a central force field in which dissipative (drag) forces are present, and where the initial motion of the body is that corresponding to equilibrium on a circular orbit at distance r from the center of the earth. In general, since the differential equations of this type of motion cannot be integrated in closed form, it is necessary to resort to approximate methods. The analysis presented here is essentially that given by Chien[15]. The equations of motion in polar coordinates referred to non-rotating axes with origin at the center of the earth are

$$\dot{v} = -g\frac{\dot{r}}{v} - \frac{D}{m}, \text{ and} \tag{173}$$

$$r^2\dot{\phi}^2 - r\dot{\phi}\ddot{r} + r\dot{r}\ddot{\phi} + 2\dot{r}^2\dot{\phi} = gr\dot{\phi} \quad , \tag{174}$$

where D is the drag, v is the total velocity in the direction of the motion, r and ϕ are polar coordinates of position, and time derivatives are indicated by the dot notation. The total velocity v must also satisfy the relation

$$v^2 = \dot{r}^2 + (r\dot{\phi})^2 \quad . \tag{175}$$

Letting v_0, ω_0, and r_0 denote respectively the linear velocity, angular velocity, and radial distance of a stationary circular orbit, we have the relation

$$\frac{v_0^2}{r_0} = g_0 = g_R\left(\frac{R}{r_0}\right)^2 \tag{176}$$

for the equilibrium of forces in the radial direction, and the velocity relation

$$\omega_0 = \frac{v_0}{r_0}. \tag{177}$$

Since D/m is very small compared to the other forces present, the actual motion will be considered as a perturbation of the stationary circular orbit, and it is then permissible to use the approximation

$$
\begin{aligned}
v &= v_0 + \frac{D}{m}\Delta v_1 + \left(\frac{D}{m}\right)^2 \Delta v_2 + \cdots \\
\dot{\phi} &= \omega_0 + \frac{D}{m}\Delta\omega_1 + \left(\frac{D}{m}\right)^2 \Delta\omega_2 + \cdots \\
r &= r_0 + \frac{D}{m}\Delta r_1 + \left(\frac{D}{m}\right)^2 \Delta r_2 + \cdots
\end{aligned}
\tag{178}
$$

where Δv_1, Δv_2, $\Delta\omega_1$, $\Delta\omega_2$, Δr_1, etc., are functions of time only and D/m is independent of time. Using only the first approximation terms Δv_1, Δr_1, $\Delta\omega_1$ of the expansion (178) and making use of relations (173), (174), and (175) it is found, by equating coefficients of the terms in D/m, that

$$
\begin{aligned}
\Delta\dot{v}_1 &= -\left(1 + g_0\frac{\Delta\dot{r}_1}{v_0}\right), \\
2\Delta r_1 r_0\omega_0^3 + 3\omega_0^2 r_0^2\Delta\omega_1 - r_0\omega_0\Delta\ddot{r}_1 &= g_0 r_0\left(\Delta\omega_1 - \Delta r_1\frac{\omega_0}{r_0}\right), \text{ and} \\
2v_0\Delta v_1 &= 2r_0\Delta r_1\omega_0^2 + 2r_0^2\omega_0\Delta\omega_1 .
\end{aligned}
\tag{179}
$$

Using relations (176) and (177), these equations may be written in the simpler form

$$
\begin{aligned}
\Delta\dot{v}_1 &= -(1 + \omega_0\Delta\dot{r}_1), \\
\Delta\ddot{r}_1 &= 3\omega_0^2\Delta r_1 + 2v_0\Delta\omega_1, \text{ and} \\
\Delta v_1 &= \omega_0\Delta r_1 + r_0\Delta\omega_1 ,
\end{aligned}
\tag{180}
$$

giving three equations in the three unknowns, Δv_1, Δr_1, $\Delta\omega_1$. Since the initial conditions at the beginning of the motion are assumed to correspond to those of the stationary circular orbit having the values v_0, r_0, ω_0, it follows that $\Delta v_1 = \Delta r_1 = \Delta\omega_1 = 0$ when $t = 0$. Equations (180) may be readily integrated, yielding the result

$$
\begin{aligned}
\Delta v_1 &= \frac{1}{\omega_0}(\omega_0 t - 2\sin\omega_0 t), \\
\Delta r_1 &= \frac{2}{\omega_0^2}(\sin\omega_0 t - \omega_0 t), \\
\Delta\omega_1 &= \frac{1}{r_0\omega_0}(3\omega_0 t - 4\sin\omega_0 t).
\end{aligned}
\tag{181}
$$

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